

TWISTED GROMOV AND LEFSCHETZ INVARIANTS ASSOCIATED WITH BUNDLES

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ABSTRACT. Given a closed symplectic 4-manifold (X, ω) , we define a twisted version of the Gromov-Taubes invariants for (X, ω) , where the twisting coefficients are induced by the choice of a surface bundle over X . Given a fibered 3-manifold Y , we similarly construct twisted Lefschetz zeta functions associated with surface bundles: we prove that these are essentially equivalent to the Jiang's Lefschetz zeta functions of Y , twisted by the representations of $\pi_1(Y)$ that are induced by monodromy homomorphisms of surface bundles over Y . This leads to an interpretation of the corresponding twisted Reidemeister torsions of Y in terms of products of "local" commutative Reidemeister torsions. Finally we relate the two invariants by proving that, for any fixed closed surface bundle \mathcal{B} over Y , the corresponding twisted Lefschetz zeta function coincides with the Gromov-Taubes invariant of $S^1 \times Y$ twisted by the bundle over $S^1 \times Y$ naturally induced by \mathcal{B} .

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1. INTRODUCTION

To any given closed symplectic 4-manifold (X, ω) , endowed with an auxiliary generic ω -compatible almost complex structure J , it is possible to associate its Gromov-Taubes series $\text{GT}(X, \omega, J)$ (defined in [22]). Even if this depends a priori on the choice of J , Taubes directly showed that in fact it depends only on X and the isotopy class of ω . Furthermore he proved in [23] that if $b_2^+(X) > 1$ then $\text{GT}(X, \omega)$ is equivalent to the Seiberg-Witten invariant of X (see for example [19]), and so it depends only on the diffeomorphism class of X .

Briefly, $\text{GT}(X)$ can be defined in terms of a weighted count of certain embedded J -holomorphic surfaces C in (X, ω, J) . The weight of each C depends on a sign and its homology class $[C] \in H_2(X, \mathbb{Z})$, encoded by a formal variable $t_{[C]}$.

In the standard case, $\text{GT}(X)$ is related to other invariants. Of particular interest for us is the case of symplectic 4-manifolds of the form $S^1 \times Y$: by a result of Friedl and Vidussi ([7]) Y must be a fibered 3-manifold. In this case it is possible to directly prove (see for example Section 2.6 of [13]) that $\text{GT}(S^1 \times Y)$ coincides with the *Lefschetz zeta function* $\zeta(Y)$, which in turn is equivalent to the Reidemeister

Date: June 30, 2016

Key words: Gromov invariants, surface bundles, Lefschetz zeta functions, twisted Reidemeister torsion.

The author was supported by ERC LTDBud.

torsion of Y (also called Milnor or Turaev torsion), which is a topological invariant of Y . We remark that if $L \subset S^3$ is a link, then the Reidemeister torsion of $S^3 \setminus L$ is essentially equivalent to the Alexander polynomial of L .

There are several refinements of the Lefschetz zeta function. One of them is the *twisted Lefschetz zeta function* $\zeta_\rho(Y)$, defined by Jiang ([16], [17]), which is associated to a representation $\rho : \pi_1(Y) \rightarrow GL(n, R)$, where R is a commutative ring with unit. Jiang proved that if Y is fibered, then $\zeta_\rho(Y)$ is equivalent to *Lin's ρ -twisted Reidemeister torsion of Y* (see for example [18]). Roughly speaking, if Y is diffeomorphic to the mapping torus of a surface S with a diffeomorphism ϕ , $\zeta_\rho(Y)$ is a weighted count of periodic orbits of ϕ , where the weight of an orbit δ depends on its Lefschetz sign, a formal variable encoding the period of δ and the “non-commutative” weight $\text{tr}(\rho([\delta]))$, where $[\delta]$ is the conjugacy class in $\pi_1(Y)$ determined by δ .

The main motivations behind this paper are the following questions:

Question 1. There exist “twisted versions” of the Gromov-Taubes invariants that are related to the twisted Reidemeister torsions like the standard $\text{GT}(S^1 \times Y)$ is related to the standard Reidemeister torsion of the fibered manifold Y ?

Question 2. There is a purely topological interpretation of the twisting coefficients of Jiang and Lin? Said differently, what is the topological meaning of the information carried by $\zeta_\rho(Y)$ for a given choice of ρ ?

In order to try to answer to these (apparently unrelated) questions we introduce the concept of “bundled twistings” for both Gromov-Taubes invariants and Lefschetz zeta functions.

We start in Section 2 by briefly recalling the definition of the standard Gromov-Taubes invariants.

In Section 3 we define a *bundled twisted version of* $\text{GT}(X, \omega, J)$, where the aim of the twisting coefficient associated to a J -holomorphic curve $C \subset X$ is to detect non-commutative informations about the homotopy class of C in X . Roughly speaking, a bundled twisting is obtained by choosing a smooth surface bundle

$$F \hookrightarrow W \xrightarrow{\pi} X$$

and then twisting the weight of C in $\text{GT}(X, \omega, J)$ by the Gromov-Taubes invariant $\text{GT}(\pi^*C)$ of the pull-back bundle over C , computed with respect to formal variables encoding the homology classes in the image of $H_2(\pi^*C, \mathbb{Z}) \xrightarrow{i_*} H_2(W, \mathbb{Z})$, where $i : \pi^*C \hookrightarrow W$ is the inclusion. Since the diffeomorphism type of π^*C depends on the isotopy class of C in W , it is natural to expect to get a refinement of the standard $\text{GT}(X, \omega, J)$, where only the homology class of C is taken into account.

What we get is a *π -twisted Gromov-Taubes series* $\text{GT}_\pi(X, \omega, J)$, depending, a priori, on the symplectic form ω and the ω -compatible almost complex structure J on X . In Subsection 3.4 we will then prove the following:

Theorem 1.1. *Given a smooth oriented surface bundle (W, X, π, F) with F closed, the π -twisted Gromov-Taubes series $\text{GT}_\pi(X, \omega, J)$ is independent on a generic choice of J and depends only on the isotopy class of ω .*

In Section 4 we apply the idea of the bundled twisting coefficients to define *π -twisted Lefschetz zeta functions* $\zeta_\pi(Y)$ for a fibered 3-manifold Y , where now π is the projection map of a smooth surface bundle

$$F \hookrightarrow V \xrightarrow{\pi} Y.$$

The definition of $\zeta_\pi(Y)$ is conceptually similar to that of $\text{GT}_\pi(X, \omega)$: morally, the latter is defined by counting holomorphic curves in the pull-back bundles over J -holomorphic curves, while the former is defined by counting periodic orbits in the pull-back bundles over periodic orbits. The key fact is that the pull-back bundle of π over any periodic orbit is again a mapping torus with fiber F and first return map determined by the *monodromy homomorphism*

$$\rho_\pi : \pi_1(Y) \longrightarrow \text{MCG}(F)$$

of the bundle: this “local” mapping torus has again a standard Lefschetz zeta function, that can be used as twisting weight for the underlying orbit in Y .

It turns out that our π -twisted Lefschetz zeta function is essentially equivalent to the Jiang’s Lefschetz zeta function, “algebraically” twisted with respect to the matrix representations

$$\rho_{\pi*} : \pi_1(Y) \longrightarrow \text{GL}(H_*(F))$$

induced in homology by ρ_π (see Theorem 4.15 below). As a corollary we obtain that for any surface bundle (V, Y, π, F) , $\zeta_\pi(Y)$ is a *topological invariant*. In fact, since $\zeta_\pi(Y)$ is defined by counting orbits (in the total space V) just with sign and their homological weight, it can be seen as a purely geometrical interpretation of the corresponding twisted Reidemeister torsion of Lin. The intuitive idea behind this phenomenon is the following: using Lefschetz fixed point theorem, we can interpret the Jiang’s twisting coefficients $\text{tr}(\rho_{\pi*}([\delta^n]))$ of the iterates of an orbit δ as a signed sum of periodic points (orbits) in the mapping torus $\pi^*\delta$ of $(F, \rho_\pi([\delta]))$, which in turn gives (a version of) the Lefschetz zeta function of $\pi^*\delta$. This provides an answer to Question 2 for the family of *bundled representations* of $\pi_1(Y)$ of Definition 4.19 below (cf. Example 4.20 and Remark 4.23).

The reason for which the two motivational questions above are related is explained by the following:

Theorem 1.2. *Let Y be the mapping torus of a closed surface S and a diffeomorphism ϕ and let (V, Y, π, F) be a smooth oriented surface bundle with F closed. Let*

$$(S^1 \times V, S^1 \times Y, \text{Id} \times \pi, F)$$

be the natural bundle induced by the product by S^1 and consider the symplectic form $\omega = \Omega + ds \wedge dt$ on $S^1 \times Y$, where Ω is any symplectic form on S and t and s are coordinates for $[0, 1]$ and, respectively, S^1 . Then:

$$\text{GT}_{\text{Id} \times \pi}(S^1 \times Y, \omega) = \zeta_\pi(Y).$$

By last theorem, and the relation between ζ_π and Lin’s twisted Reidemeister torsion, $\text{GT}_\pi(X, \omega)$ provides an answer to Question 1 and can be considered as a “non-triviality” result for the twisted Gromov-Taubes invariants. Moreover, since twisted Reidemeister torsions are topological invariants, it is natural to ask whether the twisted GT’s are in fact in general independent also on the choice of the symplectic form and how powerful they are in detecting generic closed symplectic 4-manifolds.

We remark that twisted Reidemeister torsions are in general much stronger invariants than the standard version. For example it has been proved that they detect fibered (non-necessarily closed) 3-manifolds ([7]), Thurston norms and genus of knots ([9]) and, very recently, the hyperbolic volume of knot exteriors ([11]). Also, they give useful tools in detecting sliceness and knot concordance: see [9] for these and

other applications. It is then natural to expect to find several useful applications also for the twisted Gromov-Taubes invariants.

Acknowledgements. The author is grateful to Paolo Ghiggini and András Stipsicz for many helpful suggestions and discussions. He also would like to thank Antonio Alfieri, Vincent Colin and Chris Wendl for the interest they showed about this work.

2. REVIEW OF THE GROMOV-TAUBES INVARIANTS

In this brief review of $\text{GT}(X, \omega)$ we will follow [13, Chapter 2], [14] and [22].

Fix (X, ω) and an ω -compatible almost complex structure J over X . A J -holomorphic curve in (X, J) is a smooth map $u : (\Sigma, j) \rightarrow (X, J)$ from a connected compact Riemann surface (Σ, j) such that

$$\bar{\partial}f := \frac{1}{2}(du \circ j - J \circ du) = 0$$

and considered up to the following equivalence relation: given $u_i : (\Sigma_i, j_i) \rightarrow (X, J)$, $i = 1, 2$, then u_1 and u_2 are equivalent if there exists a biholomorphism $\phi : (\Sigma_1, j_1) \rightarrow (\Sigma_2, j_2)$ such that $u_2 \circ \phi = u_1$. We will usually denote the image of u by C_u .

If the holomorphic curve is connected it can be of two types:

- *somewhere injective*, i.e. $\exists x \in \Sigma$ such that $u^{-1}(u(x)) = \{x\}$ and $d_x u$ is injective;
- *multiply covered*, i.e. u factors through a degree bigger than 1 branched cover over another holomorphic curve.

One can show that in the first case C_u is an embedded surface outside a finite number of points, called *singularities*.

Holomorphic curves in dimension 4 have some nice properties. One of them is the *positivity of intersection*, which can be formulated as follows. Let u and v be two distinct holomorphic curves in (X, J) . If $P \in C_u \cap C_v$, then its contribution m_P to the algebraic intersection number $C_u \cdot C_v$ is strictly positive, and $m_P = 1$ if and only if u and v are embeddings near P that intersect transversely in P .

Note that the assumption that u_1 and u_2 are distinct is crucial: for example one can have embedded holomorphic spheres with negative autointersection.

Another relevant property is the following *adjunction formula*. Let $u : (\Sigma, j) \rightarrow (X, J)$ be a somewhere injective J -holomorphic curve. Then:

$$(2.1) \quad \langle c_1(TX), [C_u] \rangle = \chi(\Sigma) + [C_u] \cdot [C_u] - 2\delta(C_u)$$

where δ is a weighted count of the singularities of C_u . In particular C_u must have genus

$$(2.2) \quad g_{C_u} = 1 + \frac{1}{2}([C_u] \cdot [C_u] - \langle c_1(TX), [C_u] \rangle - \delta(C_u)).$$

Definition 2.1. Given $A \in H_2(X, \mathbb{Z})$, we define g_A to be the genus of any embedded J -holomorphic curve in the homology class A as given by the adjunction formula:

$$g_A := 1 + \frac{1}{2}(A \cdot A - \langle c_1(TX), A \rangle).$$

In the definition of the Gromov-Taubes invariants a special role will be played by the homology classes A with $g_A = 0$ and $g_A = 1$. We define then the following subsets of $H_2(X, \mathbb{Z})$:

$$\begin{aligned}\mathcal{S}_2(X) &:= \{A \in H_2(X, \mathbb{Z}) \mid g_A = 0, A \cdot A = 0\}; \\ \mathcal{T}_2(X) &:= \{A \in H_2(X, \mathbb{Z}) \mid g_A = 1, A \cdot A = 0\}.\end{aligned}$$

To any $A \in H_2(X, \mathbb{Z})$ we can also associate the following crucial number:

$$d_A := A \cdot A + \langle c_1(TX), A \rangle.$$

This (even) number coincides with the expected dimension of the moduli space $\mathcal{M}_A(X, J)$ of embedded holomorphic curves in the homology class A , so that, whenever $d_A = 0$, we are able to count the elements of $\mathcal{M}_A(X, J)$. In general, if $d_A \geq 0$ it is possible to cut the dimension of $\mathcal{M}_A(X, J)$ by restricting the attention to the 0-dimensional subspace $\mathcal{M}_A(X, J, \Omega_{d_A})$ of $\mathcal{M}_A(X, J)$ consisting of those holomorphic curves that pass through a fixed set Ω_{d_A} of $d_A/2$ generic points in X . Obviously if $d_A = 0$ then $\mathcal{M}_A(X, J, \Omega_0) = \mathcal{M}_A(X, J, \emptyset) = \mathcal{M}_A(X, J)$.

2.1. Definition of GT.

The Gromov-Taubes invariant can be defined as a weighted count of the elements of the moduli spaces $\mathcal{M}_A(X, J, \Omega_{d_A})$ with $A \in H_2(X, \mathbb{Z})$ such that $d_A \geq 0$; the weight of each embedded J -holomorphic curve is defined in terms of:

- (1) the sign $\epsilon(C_u)$ of the determinant of a differential operator associated to each C_u (see [20] or [22] for more details);
- (2) formal variables $t_A, A \in H_2(X)$ keeping track of the homology class of the curves and satisfying $t_{A+B} = t_A \cdot t_B$.

The invariant can then be seen as a series

$$\text{GT}(X) = \sum_A \text{GT}(X, A) t_A \in \mathbb{Z}[[H_2(X)]].$$

Definition 2.2. For any $A \in H_2(X)$, let $\mathcal{D}(A)$ be the set of couples $(A_i, n_i) \in H_2(X) \times \mathbb{N}^*$ such that:

- (1) $A = \sum_i n_i A_i$;
- (2) $d_{A_i} \geq 0$;
- (3) the A_i 's are distinct and not *multiply toroidal* (i.e. of the form mB with $m \geq 2$ and $g_B = 1$);
- (4) $A_i \cdot A_j = 0$ when $i \neq j$;
- (5) if $A_i \cdot A_i \neq 0$ then $n_i = 1$;
- (6) if $A_i \cdot A_i = 0$ then n_i can be any positive integer.

Observe that if there exists a holomorphic curve in a class A_i with $d_{A_i} \geq 0$ and $A_i \cdot A_i = 0$, then the adjunction formula implies that A_i is either in $\mathcal{S}_2(X)$ or in $\mathcal{T}_2(X)$. Note moreover that $t_A = \sum_i t_{n_i A_i} = \sum_i t_{A_i}^{n_i}$ and $d_A = \sum_i n_i d_{A_i}$ (cf. [13, Lemma 2.4]).

Now, given A , fix $d_A/2$ generic points in X ; $\text{GT}(X, A)$ is then of the form:

$$\text{GT}(X, A) = \sum_{(A_i, n_i) \in \mathcal{D}(A)} d_A! \left(\prod_{A_i \notin \mathcal{T}_2(X)} \frac{\mathfrak{Ru}(A_i)^{n_i}}{d_{A_i}! n_i!} \cdot \prod_{A_i \in \mathcal{T}_2(X)} \Omega u(A_i, n_i) \right).$$

Note that in the first product if $n_i \neq 1$ then $A_i \in \mathcal{S}_2(X)$. For any $B \in H_2(X)$, the quantity $\mathfrak{Ru}(B)$ is the *Ruan invariant* of (X, B) ([20]) and it is defined by:

$$(2.3) \quad \Re u(B) := \sum_{C_u \in \mathcal{M}_B(X, J, \Omega_{dB})} \epsilon(C_u).$$

It is important here to recall that each of the C_u is embedded and connected by definition.

The definition of $\mathfrak{Q}u(B, n)$ for a (non multiply) toroidal class $B \in \mathcal{T}_2(X)$ is a bit more complicated. It can be expressed in the form

$$(2.4) \quad \mathfrak{Q}u(B, m) := \sum_{\{(C_k, m_k)\}} \prod_k r(C_k, m_k)$$

where the sum is over the sets of couples $(C_k, m_k) \in \mathcal{M}_{c_k B}(X, J) \times \mathbb{N}^*$ with $c_k \geq 1$ and $\sum_k c_k m_k = m$. Observe that $B \in \mathcal{T}_2(X)$ if and only if $B \cdot B = \langle c_1(TX), B \rangle = 0$, so that also $cB \in \mathcal{T}_2(X)$ for any $c \geq 1$. Then all the C_k are (embedded) tori.

If C is an embedded holomorphic torus C with homology class in $\mathcal{T}_2(X)$, in [22] Taubes extends the definition of the sign ϵ also to the holomorphic double covers of C . The numbers $r(C, l)$ can be defined in terms of the *Taubes' generating functions* $P(C, z)$: these depend on the signs of C and of its three connected double covers, which can be identified with the cohomology classes $\iota_1, \iota_2, \iota_3 \in H^1(C, \mathbb{Z}/2)$.

We say that C is of *type* $(\epsilon, s) \in \{\pm\} \times \{0, 1, 2, 3\}$ if $\epsilon(C) = \epsilon$ and exactly s of the connected double covers $\iota_1, \iota_2, \iota_3$ have sign $-$. We define then

$$P(C, z) := P_{\epsilon(C), s}(z).$$

where the functions $P_{\pm, s}(z)$ are defined in terms of $P_{+, 0}(z)$ by

$$(2.5) \quad \begin{aligned} P_{+, 1}(z) &:= \frac{P_{+, 0}(z)}{P_{+, 0}(z^2)} & , & \quad P_{+, 2}(z) := \frac{P_{+, 0}(z)P_{+, 0}(z^4)}{(P_{+, 0}(z^2))^2} , \\ P_{+, 3}(z) &:= \frac{P_{+, 0}(z)P_{+, 0}(z^4)}{(P_{+, 0}(z^2))^3} & , & \quad P_{-, s}(z) := \frac{1}{P_{+, s}(z)} \end{aligned}$$

and then normalized by setting

$$(2.6) \quad P_{+, 0}(z) := \frac{1}{1 - z}.$$

In Section 3.4 we will say something about where these generating functions come from.

For any l , the number $r(C, l)$ is then defined as the coefficient of z^l in the formal power series expansion (about 0) of $P(C, z)$. Then $\mathfrak{Q}u(B, m)$ in Equation (2.4) is the coefficient of t_{mB} in

$$\prod_{C \text{ embedded} \mid [C] \in \mathcal{T}_2(X)} P(C, t_{[C]}).$$

Remark 2.3. Observe that in the definition of Taubes' generating functions there are terms that depend on z , z^2 and z^4 : intuitively the term in z^i counts (with signs) i -fold covers of the tori. Note moreover that using these functions, for any torus C we get $r(C, 1) = \epsilon(C)$ as expected.

2.2. About the symplectic invariance of GT.

The definition of GT depends strongly on the choice of the symplectic form ω , the ω -compatible almost complex structure and the sets Ω_d . In Sections 4 and 5 of [22], Taubes proves that, for a fixed ω , there exists a dense open subset \mathcal{U}_d of the set of the “admissible” couples $\mathcal{A}_d \subset \{(J, \Omega_d)\}$ such that $\mathcal{M}_A(X, J, \Omega_d)$ (and so also both \mathfrak{Ru} and \mathfrak{Qu}) is independent on (J, Ω_d) whenever (J, Ω_d) varies within a small enough neighborhood of any point of \mathcal{U}_d . Moreover (see assertion 3 of [22, Proposition 5.2]):

Lemma 2.4. *For any smooth path $\{\omega_t \mid t \in [0, 1]\}$ of symplectic structures on X , there exists a smooth path $\{J_t \mid t \in [0, 1]\}$ of ω_t -compatible almost complex structures (with prescribed J_0 and J_1) such that the fibered product*

$$(2.7) \quad \mathcal{C}_A(X, \{J_t\}, \Omega_d) := \{(t, C_t) \mid t \in [0, 1], C_t \in \mathcal{M}_A(X, J_t, \Omega_d)\}$$

has the structure of 1-dimensional oriented manifold, which is also compact if A is not multiply toroidal.

This is enough to conclude that \mathfrak{Ru} is a symplectic invariant.

To prove that also \mathfrak{Qu} is a symplectic invariant, Taubes needed to manage the case of multiply toroidal classes. The issue with these classes is compactness: if $A \in \mathcal{T}_2(X)$ is not multiply toroidal and m is a positive integer, then it can happen that a sequence of embedded tori in $\mathcal{C}_{mA}(X, \{J_t\})$ has no limit in $\mathcal{C}_{mA}(X, \{J_t\})$ (we avoid to refer to $\Omega_0 = \emptyset$ in the notation). To solve the problem of the sequences that have no “true” limits, for any fixed n , Taubes considers the set

$$(2.8) \quad \mathcal{K}_{nA}(X, \{J_t\}) := \bigcup_{m \leq n} \mathcal{C}_{mA}(X, \{J_t\})$$

endowed with the topology induced by the disjoint union. Note that, by definition, every point in $\mathcal{K}_{nA}(X, \{J_t\})$ corresponds to an *embedded torus* whose homology class is some multiple of A . Taubes proves then the following Lemma ([22, Lemma 5.8]).

Lemma 2.5. *Even if a sequence $\{(t, C_t)\}_{t \rightarrow t_0}$ in $\mathcal{C}_{mA}(X, \{J_t\})$ has no limit in $\mathcal{C}_{mA}(X, \{J_t\})$, (t, C_t) still has a “weak limit” in $\mathcal{K}_{nA}(X, \{J_t\})$ for any $n \geq m$, i.e. there exist p and q positive integers with $pq = m$ such that $\{C_t\}_{t \rightarrow t_0}$ converges to a p -fold holomorphic cover of a torus belonging to $\mathcal{M}_{qA}(X, J_{t_0}) \subset \mathcal{C}_{qA}(X, \{J_t\}) \subset \mathcal{K}_{nA}(X, \{J_t\})$.*

If the path $\{\omega_t \mid t \in [0, 1]\}$ is “reasonable”, Taubes proves that we can have only weak limits with $p = 2$ and that the total number of the corresponding “bifurcation points” (t_0, C_{t_0}) in $\mathcal{K}_{nA}(X, \{J_t\})$ is finite: here is where the double covers of the tori come into play (see Lemmas 5.8-5.11 in [22]). Taubes studies what happens at each bifurcation point to the signs of the tori and of their double covers and finds that, in order to produce a symplectic invariant by counting tori C , their weights $P(C)$ must satisfy certain relations induced by the path $\{\omega_t\}$ when it crosses bifurcation points. These relations are exactly those in 2.5. The normalization (2.6) for $P_{+,0}$ has then been chosen by Taubes to make GT coincide with the Seiberg-Witten invariants.

3. TWISTED GROMOV-TAUBES INVARIANTS

Fix a closed symplectic 4-manifold (X, ω) endowed with an ω -compatible almost complex structure J and fix a smooth fiber bundle

$$(3.1) \quad F \hookrightarrow W \xrightarrow{\pi} X$$

where F is a closed oriented surface. Our aim is to define an analogue of $\text{GT}(X, \omega)$ in which the weight of each J -holomorphic embedded curve $C \subset X$ is (morally) twisted by the Gromov-Taubes invariant $\text{GT}(\pi^*C)$ of the total space π^*C of the restriction of π to C (here we prefer to use the notation π^*C instead of $\pi^{-1}(C)$).

Theorem 3.1 (Thurston, [24]). *Given a smooth surface bundle*

$$F \hookrightarrow N \xrightarrow{\pi} M$$

over a $2n$ -dimensional symplectic manifold (M, ω) , if the homology class of the fiber in $H_2(N, \mathbb{R})$ is non-zero, then there exists a closed 2-form α on N that is non-singular on each fiber and such that

$$(3.2) \quad \alpha + \pi^*\omega \in \Omega^2(N)$$

is symplectic.

Remark that the 2-form in (3.2) makes all the fibers symplectic and that the condition $[F] \neq 0$ in $H_2(N, \mathbb{R})$ is always satisfied when the genus of F is greater than 1. When the dimension of the base M is 2 we have also the following:

Lemma 3.2. *Assume that in the last theorem $n = 1$, so that (M, ω) is a symplectic surface. Let ω_N and ω'_N be two symplectic forms on N like in (3.2). Then the homotopy classes of the ω_N - and ω'_N -compatible almost complex structures coincide.*

Proof. Since the fibers are symplectic with respect to ω_N and ω'_N and any two positive symplectic forms on an oriented surface are deformation equivalent, we can find ω_N - and, respectively, ω'_N -compatible almost complex structures J_N and J'_N whose restriction to the tangent space of the fiber $T_x F \subset T_x N$ coincide for any $x \in N$. Let O_x and O'_x be the orthogonal complements of $T_x F$ in $T_x N$ with respect to the Riemannian metrics $\omega_N(\cdot, J\cdot)$ and, respectively, $\omega'_N(\cdot, J'\cdot)$. These give two distributions O and O' of (J - and, respectively, J' -holomorphic) tangent planes in TN , which are homotopic since the space of Riemannian metrics over N is contractible. This implies that there is a homotopy between the two decompositions $TF \oplus O$ and $TF \oplus O'$ of TN , which sends J to J' . \square

In order to define the π -twisted Gromov-Taubes invariants, we will define first π -twisted versions of \mathfrak{Ru} and \mathfrak{Qu} . For simplicity, from now on and if not stated otherwise, all the homology groups will be considered with coefficients in \mathbb{Z} . Moreover we will often implicitly assume that our surface bundles have non-trivial homology class of the fiber.

3.1. Twisted \mathfrak{Ru} .

Notation 3.3. If M is a manifold, $L \subset M$ a submanifold and $i \in \mathbb{N}$, given a homology class $A \in H_i(L)$ we will call A_M its image in $H_i(M)$ under the homomorphism induced by the inclusion $L \hookrightarrow M$. Moreover, given a group $(G, +)$ and a ring R , $R[[G]]$ will denote the polynomial ring $R[[\{t_g | g \in G\}]]$ with the usual relations $t_g \cdot t_{g'} = t_{g+g'}$. When G will be given as a direct sum $G = H \oplus I$, in order to keep distinct the variables associated to elements in H and I , $\mathbb{Z}[[G]] = \mathbb{Z}[[H \oplus I]]$ should be thought of as $R[[\{t_h | h \in H\}]] [[\{t_i | i \in I\}]]$.

For the rest of this section we fix a closed symplectic 4-manifold (X, ω) , an ω -compatible almost complex structure J and a smooth surface bundle (W, X, π, F) with $[F] \neq 0 \in H_2(W, \mathbb{R})$.

Given a surface $C \subset X$, consider the 4-manifold π^*C . The natural inclusion $\pi^*C \hookrightarrow W$ induces in homology the homomorphism $B \mapsto B_W$, where $B \in H_2(\pi^*C)$. Considering formal variables z_D , for $D \in H_2(W)$, analogue to the t_A 's of last section, we have then the ring homomorphism

$$\begin{aligned} \mathbb{Z}[[H_2(\pi^*C)]] &\longrightarrow \mathbb{Z}[[H_2(W)]] \\ at_B &\longmapsto az_{B_W} \end{aligned}$$

Now, if $C \subset X$ is a J -holomorphic embedded surface, $\omega|_C$ is symplectic and by Theorem 3.1 there exists a closed $\alpha_C \in \Omega^2(\pi^*C)$ such that the 2-form

$$(3.3) \quad \omega_C = \alpha_C + \pi^*\omega|_C \in \Omega^2(\pi^*C)$$

is symplectic and makes the fibers symplectic as well. We can then consider the Gromov-Taubes invariants

$$(3.4) \quad \text{GT}(\pi^*C, \omega_C) = \sum_{B \in H_2(\pi^*C)} \text{GT}(\pi^*C, \omega_C, B) t_B \in \mathbb{Z}[[H_2(\pi^*C)]]$$

By the statement of the equivalence between Gromov-Taubes and Seiberg-Witten invariants, $\text{GT}(\pi^*C, \omega_C)$ depends on ω_C only through the Spin^c -structure of an ω_C -compatible almost complex structure (see [23]). Lemma 3.2, implies then that $\text{GT}(\pi^*C, \omega_C)$ does not depend on the particular ω_C given by Theorem 3.1.

Definition 3.4. Given (W, X, π, F) and a J -holomorphic embedded surface $C \subset X$ we define

$$\text{GT}_W(\pi^*C) := \sum_{B \in H_2(\pi^*C)} \text{GT}(\pi^*C, \omega_C, B) z_{B_W} \in \mathbb{Z}[[H_2(W)]]$$

where ω_C is any symplectic form like in (3.3).

Observe that $\text{GT}_W(\pi^*C)$ depends only on the isotopy class of C in X . Now, for any $d \in \mathbb{N}$, fix a set Ω_d of $d/2$ generic points in X and, for a homology class A , let $\mathcal{M}_A(X, J, \Omega_{d_A})$ be as in the definition of the standard $\mathfrak{Ru}(A, \omega, J, \Omega_{d_A})$.

Definition 3.5. Given the bundle (W, X, π, F) and a class $A \in (H_2(X) \setminus \mathcal{T}_2(X))$, we define the π -twisted Ruan invariant of (X, A) by:

$$(3.5) \quad \mathfrak{Ru}_\pi(A, \omega, J, \Omega_{d_A}) := \sum_{C \in \mathcal{M}_A(X, J, \Omega_{d_A})} \epsilon(C_u) \cdot \text{GT}_W(\pi^*C).$$

When ω , J and Ω_d are understood we will simply write $\mathfrak{Ru}_\pi(A)$ for $\mathfrak{Ru}_\pi(A, \omega, J, \Omega_d)$.

3.2. Twisted Ωu .

As in the standard case, the count for homology classes in $\mathcal{T}_2(X)$ is more complicated. As recalled in last section, the weight in $\text{GT}(X)$ for a J -holomorphic embedded torus C with $[C] \in \mathcal{T}_2(X)$ depends on the signs $\epsilon(C_{\iota_i})$ of its connected double covers C_{ι_i} , $i \in \{1, 2, 3\}$.

We want to define π -twisted analogues of the Taubes' generating functions in (2.5). To do that it will be convenient to have an explicit description of the double covers ι_i of C . Identify C with $\mathbb{R}^2/\mathbb{Z}^2$ and fix the basis of $H_1(C, \mathbb{Z})$ induced by the ordered couple of segments $([0, 1] \times \{0\}, \{0\} \times [0, 1])$ of \mathbb{R}^2 quotiented by \mathbb{Z}^2 . The covering spaces C_{ι_i} , $i \in \{1, 2, 3\}$, of the three relevant double covers can then be represented as the images on \mathbb{R}^2 of the square $[0, 1]^2$ under the linear maps

$$l_{\iota_1} := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad l_{\iota_2} := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad l_{\iota_3} := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

with their boundary quotiented by the relation induced by the quotient $[0, 1]^2/\mathbb{Z}^2$. Observe that the covering map $f_{\iota_i} : C_{\iota_i} \rightarrow C$ is given by the restriction to $\text{Im}(l_{\iota_i})$ of quotient of \mathbb{R}^2 by \mathbb{Z}^2 . The relevant covers can then be identified with the elements $\iota_1, \iota_2, \iota_3 \in H^1(C, \mathbb{Z}/2) \cong \text{Hom}(H_1(C, \mathbb{Z}), \mathbb{Z}/2)$ represented by the matrices

$$(3.6) \quad \iota_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \iota_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \iota_3 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

acting on the left on column vectors with respect to the fixed basis of $H_1(C, \mathbb{Z})$.

Note that the image under l_{ι_i} of the couple of segments $([0, 1] \times \{0\}, \{0\} \times [0, 1])$ induces a basis of $H_1(C_{\iota_i}, \mathbb{Z})$ and that the covering map f_{ι_i} induces in homology the map $f_{\iota_i*} : H_1(C_{\iota_i}, \mathbb{Z}) \rightarrow H_1(C, \mathbb{Z})$, which, with respect to the fixed bases, is given exactly by the matrices above:

$$(3.7) \quad f_{\iota_1*} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad f_{\iota_2*} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad f_{\iota_3*} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

In Remark 2.3, we said that intuitively the terms depending on z^i are associated to i -fold covers of the tori. In particular, the terms in z^4 are associated to double covers over the C_{ι} 's, which in turn can be identified with the elements of $H^1(C_{\iota}, \mathbb{Z}/2)$.

The 4-fold covers of C that interest us are associated to pairs of double covers (ι_i, ι_j) of C , with $\iota_i \neq \iota_j$. Observe that the cover ι_i induces the map

$$f_{\iota_i*} : H_1(C_{\iota_i}, \mathbb{Z}/2) \rightarrow H_1(C, \mathbb{Z}/2)$$

as defined in (3.7) (but with $\mathbb{Z}/2$ coefficients). Then the homomorphism

$$\iota_{j \circ i} := f_{\iota_i}^*(\iota_j) = \iota_j \circ f_{\iota_i*} : H_1(C_{\iota_i}, \mathbb{Z}/2) \rightarrow \mathbb{Z}/2$$

defines a class in $H^1(C_{\iota_i}, \mathbb{Z}/2)$, and so a double cover $C_{\iota_{j \circ i}}$ of C_{ι_i} (cf. the map given in (5.22) of [22]).

Lemma 3.6. *For every pair of double covers (ι_i, ι_j) of C with $\iota_i \neq \iota_j$, the 4-fold cover*

$$f_{\iota_i} \circ f_{\iota_{j \circ i}} : C_{\iota_{j \circ i}} \rightarrow C$$

is the unique determined by the image of the square $[0, 1]^2 \subset \mathbb{R}^2$ by the matrix

$$(l_{\iota_i} \circ l_{\iota_{j \circ i}})_* = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Proof. We show only the case (ι_1, ι_3) , leaving to the reader the analogue computations for the other cases. First we have

$$\iota_{3 \circ 1} = \iota_3 \circ f_{\iota_1*} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in H^1(C_{\iota_1}, \mathbb{Z}/2).$$

Then $\iota_{3 \circ 1}$ is the double cover ι_2 of C_{ι_1} and:

$$(l_{\iota_1} \circ l_{\iota_{3 \circ 1}})_* = f_{\iota_1*} \circ f_{\iota_2*} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

□

Observe in particular that the lemma implies that the 4-fold covers of C determined by (ι_i, ι_j) and (ι_j, ι_i) are the same. We will denote by ι_4 the 4-fold cover of C given by last lemma and by $f_{\iota_4} : C_{\iota_4} \rightarrow C$ the corresponding covering map.

When we defined GT_W we got a polynomial with formal variables z_{B_W} , where the B_W 's are homology classes in W induced by homology classes B on the pull-back bundles over the holomorphic curves. In the case of tori it will be convenient to keep track of the projection on X of the classes B_W .

Definition 3.7. Let $C \subset X$ be an embedded torus. We define

$$\mathrm{GT}_X(\pi^*C) := \sum_{B \in H_2(\pi^*C)} \mathrm{GT}(\pi^*C, \omega_C, B) z_{B_W} t_{\pi_*(B_W)} \in \mathbb{Z}[[H_2(W) \oplus H_2(X)]],$$

where ω_C is any symplectic form like in (3.3).

Observe that the informations contained in $\mathrm{GT}_X(\pi^*C)$ are exactly the same as $\mathrm{GT}_W(\pi^*C)$ (as defined in last subsection), which can be recovered just by setting $t_A = 1$ for all $A \in H_2(X)$. It is convenient to regard $\mathrm{GT}_X(\pi^*C)$ as a series in the variables t_A and coefficients that are series in the variables z_B . Note finally that, by definition, if the coefficient of t_A in $\mathrm{GT}_X(\pi^*C)$ is non-zero, then A is a non-negative multiple of $[C]$.

Now, given an embedded torus $C \subset X$ with $[C] \in \mathcal{T}_2(X)$ and one of the four relevant covers $\iota : C_\iota \rightarrow C$ above, let us denote by $\pi^*C_\iota := f_\iota^*(\pi^*C)$ the total space of the bundle over C_ι and fiber F obtained by pulling back via f_ι the bundle structure of π^*C . Then the smooth 4-manifold π^*C_ι is the total space of a (2- or 4-fold) cover of π^*C and we will call π^*f_ι the corresponding covering map.

$$\begin{array}{ccc} \pi^*C_\iota & \xrightarrow{\pi^*f_\iota} & \pi^*C \\ \pi \downarrow & & \downarrow \pi \\ C_\iota & \xrightarrow{f_\iota} & C \end{array}$$

Observe that, if ω_C is a symplectic form over π^*C given by Theorem 3.1, the pull-back $\omega_{C_\iota} := (\pi^*f_\iota)^*(\omega_C)$ is symplectic over π^*C_ι and makes the fibers symplectic.

Definition 3.8. Let $C \subset X$ be an embedded torus with $[C] \in \mathcal{T}_2(X)$ and $\iota : C_\iota \rightarrow C$ one of the covers $\{\iota_1, \iota_2, \iota_3, \iota_4\}$. Define

$$\mathrm{GT}_X(\pi^*C_\iota) := \sum_{B \in H_2(\pi^*C_\iota)} \mathrm{GT}(\pi^*C_\iota, \omega_{C_\iota}, B) z_{B_W} t_{\pi_*(B_W)} \in \mathbb{Z}[[H_2(W) \oplus H_2(X)]],$$

where, with slight abuse of notation, we set $B_W = ((\pi^*f_\iota)_*(B))_W$

In what follows, we will need to keep track not only of the type $(\epsilon(C), s) \in \{\pm 1\} \times \{0, 1, 2, 3\}$ of a torus C defined in the end of Subsection 2.1, but also of the particular s double covers of C having negative sign. We will say then that C is of type $(\epsilon(C), I)$ if $I \subseteq \{\iota_1, \iota_2, \iota_3\}$ is the set of the s covers of C that have negative sign.

Definition 3.9. Given (W, X, π, F) , let C be an embedded holomorphic torus of type $(\epsilon(C), I)$ with homology class in $\mathcal{T}_2(X)$. We define the following Laurent series in the formal variables t_A , $A \in H_2(X)$ and coefficients that are series in the variables

$z_B, B \in H_2(W)$:

$$P_\pi(C) := \begin{cases} (\text{GT}_X(\pi^*C))^{\epsilon(C)} & \text{if } I = \emptyset \\ \left(\frac{\text{GT}_X(\pi^*C)}{\text{GT}_X(\pi^*C_{\iota'})} \right)^{\epsilon(C)} & \text{if } I = \{\iota'\} \\ \left(\frac{\text{GT}_X(\pi^*C) \text{GT}_X(\pi^*(C_{\iota_4}))}{\text{GT}_X(\pi^*C_{\iota'}) \text{GT}_X(\pi^*C_{\iota''})} \right)^{\epsilon(C)} & \text{if } I = \{\iota', \iota''\} \\ \left(\frac{\text{GT}_X(\pi^*C) \text{GT}_X(\pi^*(C_{\iota_4}))}{\text{GT}_X(\pi^*C_{\iota_1}) \text{GT}_X(\pi^*C_{\iota_2}) \text{GT}_X(\pi^*C_{\iota_3})} \right)^{\epsilon(C)} & \text{if } I = \{\iota_1, \iota_2, \iota_3\} \end{cases}$$

Definition 3.10. Given $A \in \mathcal{T}_2(X)$ primitive, we define

$$(3.8) \quad \mathfrak{Qu}_\pi(A, m, \omega, J) := \sum_{\{(C_k, m_k)\}} \prod_k r_\pi(C_k, m_k)$$

where the sum is over the sets of couples $(C_k, m_k) \in \mathcal{M}_{c_k A}(X) \times \mathbb{N}^*$ with $c_k \geq 1$ and $\sum_k c_k m_k = m$ and $r_\pi(C_k, m_k)$ is defined to be the coefficient of $t_{m_k[C_k]}$ in the formal power series $P_\pi(C_k)$.

When ω and J are understood we will write $\mathfrak{Qu}_\pi(A, m)$ instead of $\mathfrak{Qu}_\pi(A, m, \omega, J)$.

Remark 3.11. Note that $\mathfrak{Qu}_\pi(A, m)$ is the coefficient of t_{mA} in

$$\prod_{C \text{ embedded} \mid [C] \in \mathcal{T}_2(X)} P_\pi(C).$$

3.3. Twisted GT.

We can now define the π -twisted Gromov-Taubes invariants exactly like in the standard case. For any $d \in \mathbb{N}$, fix $d/2$ generic points in X .

Definition 3.12. Given (W, X, π, F) , we define the π -twisted Gromov-Taubes invariant of (X, ω, J) by

$$\text{GT}_\pi(X, \omega, J) := \sum_{A \in H_2(X)} \text{GT}_\pi(X, A) t_A \in \mathbb{Z}[[H_2(W) \oplus H_2(X)]]$$

where

$$\text{GT}_\pi(X, A) = \sum_{(A_i, n_i) \in \mathcal{D}(A)} d_A! \left(\prod_{A_i \notin \mathcal{T}_2(X)} \frac{\mathfrak{Qu}_\pi(A_i)^{n_i}}{d_{A_i}! n_i!} \cdot \prod_{A_i \in \mathcal{T}_2(X)} \mathfrak{Qu}_\pi(A_i, n_i) \right)$$

and $\mathcal{D}(A)$ is the set of decompositions given in Definition 2.2.

At this point some words about the variables encoding the homology classes in W and X is due. Think for example to a holomorphic curve $L \subset \pi^*C$ with $g_{[L]} > 1$ for some embedded holomorphic curve $C \subset X$ with $g_{[C]} > 1$. Its total contribution to $\text{GT}_\pi(X, \omega, J)$ is

$$(3.9) \quad \epsilon(C) \epsilon(L) z_{[L]_W} t_{[C]},$$

where $z_{[L]_W}$ and $t_{[C]}$ are in general independent (and in particular we could have $\pi_*([L]_W) \neq [C]$). On the other hand, if $g_{[C]} = 1$, the total contribution of L is more

complicated and in general it is not even of the form $ct_{[C]}$ for some Laurent series c with variables z_B , $B \in H_2(W)$. Roughly speaking the problem is that L could “appear” also in some cover \tilde{C} of π^*C , in which case we do not know on which $t_{[\tilde{C}]}$ the contribution of L should depend: the reason for which to set $t_{\pi_*([L]_W)}$ for the $H_2(X)$ -component of the weight of L is a natural choice will appear more clear in Section 4.

3.4. Proof of the symplectic invariance.

In Subsection 2.2 we gave a short overview towards the proof of the symplectic invariance of the standard $\text{GT}(X)$. Since the signs of the weights associated to the holomorphic curves are the same of Taubes, to prove the symplectic invariance of twisted GT it will be enough to carefully analyze the purely topological part of Taubes’ proof. By this we mean the following: given (W, X, π, F) and a holomorphic curve $C \in \mathcal{M}_A(X, J, \Omega_{d_A})$, we want to understand what happens to π^*C and, eventually, its relevant covers, when we change Ω_{d_A} , J or we follow a smooth path of symplectic structures. The answer is mostly completely contained in the proofs in [22, Section 5] and [20, Sections 4 and 5]. In what follows we will use the notations introduced in Subsection 2.2.

Lemma 3.13. *For any $A \in H_2(X)$ and $(J, \Omega_d) \in \mathcal{U}_d$ (where $d = d_A$), there is an open neighborhood $U = U(J, \Omega_d)$ of (J, Ω_d) in \mathcal{U}_d such that for any $(J', \Omega'_d) \in U$, there is an orientation preserving homeomorphism of (0-dimensional compact oriented) manifolds*

$$\mathcal{M}_A(X, J, \Omega_d) \cong \mathcal{M}_A(X, J', \Omega'_d)$$

that sends any $C \in \mathcal{M}_A(X, J, \Omega_d)$ to a $C' \in \mathcal{M}_A(X, J', \Omega'_d)$ which is isotopic and close to C in X .

Lemma 3.13 is essentially the Assertion 2 of [22, Proposition 5.2]. The “isotopic” part is not explicitly stated but follows from Taubes’ proof. Briefly, $\mathcal{M}_A(X, J, \Omega_d)$ is the counter-image of a regular value (J, Ω_d) of a certain smooth function with image in \mathcal{A}_d . By the Sard-Smale theorem, the set of the regular values of this function must be an open and dense subset of \mathcal{A}_d and the inverse function theorem implies that if (J', Ω'_d) is close enough to (J, Ω_d) in \mathcal{A}_d , then there is a trivial cobordism from $\mathcal{M}_A(X, J, \Omega_d)$ to $\mathcal{M}_A(X, J', \Omega'_d)$ that gives a homotopy in X between each C to a C' . For more details see the first steps of the proof of [22, Proposition 5.2] (and also [20, Section 5]).

The fact that we can actually get an isotopy comes by further restricting the choice for the couples (J, Ω_d) . Taubes observes in fact that, if A is not multiply toroidal, the values of the time of the homotopy above where this fails to give an embedding correspond to the degenerations appearing in the Gromov compactness theorem ([12]). On the other hand, Taubes proves (see step 5 of the proof of the aforementioned Taubes’ proposition) that there exists an open and dense subset (i.e. \mathcal{U}_d) of \mathcal{A}_d for which no such degeneration can occur, and the isotopy follows for $g_{[A]} > 1$. Finally, some local degenerations (like bifurcations) that could a priori still happen if A is multiply toroidal are excluded (for U small enough) in the final steps of the proof of Taubes’ proposition.

Corollary 3.14. *Let (W, X, π, F) be a surface bundle, $A \in H_2(X)$ and $m \in \mathbb{N}$. Then $\mathfrak{Ru}_\pi(A, \omega, J, \Omega_{d_A})$ and $\mathfrak{Qu}_\pi(A, m, \omega, J)$ do not depend on the choice of (J, Ω_{d_A}) in a small neighborhood U of (J, Ω_{d_A}) in \mathcal{A}_d .*

Proof. If C and C' are embedded isotopic surfaces in X with $g_{[C]} > 1$ then the two 4-manifolds π^*C and π^*C' are isotopic in W and $\text{GT}_W(\pi^*C) = \text{GT}_W(\pi^*C')$. Observe that here we did not use only the fact that π^*C and π^*C' are diffeomorphic, but also the fact that (by Lemma 3.2 and the statement of “GT = SW”), $\text{GT}_W(\pi^*C)$ and $\text{GT}_W(\pi^*C')$ have the same normalization of the variables.

Similarly, if $g_{[C]} = 1$, it is easy to check that $\text{GT}_X(\pi^*C) = \text{GT}_X(\pi^*C')$ and, identifying the 2- and 4-fold covers using the isotopy, for any ι we have also $\text{GT}_X(\pi^*C_\iota) = \text{GT}_X(\pi^*C'_\iota)$. The result follows then from Lemma 3.13. \square

Corollary 3.15. *Let (W, X, π, F) be a surface bundle, $A \in H_2(X) \setminus \mathcal{T}_2(X)$ and $\{\omega_t \mid t \in [0, 1]\}$ a smooth path of symplectic structures on X . Then*

$$\mathfrak{Ru}_\pi(A, \omega_0) = \mathfrak{Ru}_\pi(A, \omega_1).$$

Proof. The proof is very similar to that of last corollary and uses the smooth cobordism $\mathcal{C}_A(X, \{J_t\}, \Omega_d)$ given by Lemma 2.4. The main difference is that here the cobordism may be non-trivial since the projection $\mathcal{C}_A(X, \{J_t\}, \Omega_d) \rightarrow [0, 1]$ can have critical points (other degenerations are excluded by Taubes in Step 7 of the proof of Proposition 5.2 of [22]). These critical points correspond to births or deaths of couples of holomorphic curves C_1 and C_2 with $\epsilon(C_1) = -\epsilon(C_2)$ whose total contribution, in the non-twisted case, obviously does not affect \mathfrak{Ru} . Furthermore C_1 and C_2 must be in the same connected component of $\mathcal{C}_A(X, \{J_t\}, \Omega_d)$, and this gives an isotopy in X between C_1 and C_2 so that their total contribution

$$\epsilon(C_1)\text{GT}_W(\pi^*C_1) + \epsilon(C_2)\text{GT}_W(\pi^*C_2)$$

to \mathfrak{Ru}_π is also zero, since $\text{GT}_W(\pi^*C_1) = \text{GT}_W(\pi^*C_2)$. \square

The proof of the symplectic invariance for \mathfrak{Qu}_π is more delicate since, for a given $B \in \mathcal{T}_2(X)$, the 1-dimensional manifold $\mathcal{C}_B(X, \{J_t\})$ is not in general compact. As explained in Subsection 2.2, a sequence of tori in $\mathcal{C}_{2B}(X, \{J_t\})$ could converge, for $t \rightarrow t_0$, to a 2-fold cover of a holomorphic torus in $\mathcal{M}_B(X, J_{t_0})$ (called the weak limit of the sequence).

Now, if $A \in \mathcal{T}_2(X)$ is primitive and the path $\{\omega_t \mid t \in [0, 1]\}$ is chosen carefully, then, for any $n \in \mathbb{N}$, these weak limits correspond to the bifurcation points of $\mathcal{K}_{nA}(X, \{J_t\})$ (defined in (2.8)) and these are exactly the points of $\mathcal{K}_{nA}(X, \{J_t\})$ where the latter fails to be compact (recall that $\mathcal{K}_{nA}(X, \{J_t\})$ has the topology induced by the disjoint union). Fix $n \in \mathbb{N}$ and consider the natural projection

$$\begin{aligned} v_{nA} : \mathcal{K}_{nA}(X, \{J_t\}) &\longrightarrow [0, 1] \\ (t, C) &\longmapsto t. \end{aligned}$$

By [22, Lemma 5.8], $\mathcal{K}_{nA}(X, \{J_t\})$ is a 1-dimensional manifold and we can assume that its bifurcation points and the critical points of v_{nA} (births and deaths of pairs of tori with opposite signs) are finite and the corresponding values of $t \in [0, 1]$ are all distinct and in $(0, 1)$. Let us call $t \in (0, 1)$ a *bifurcation value* of v_{nA} if $v_{nA}^{-1}(t_0)$ contains a bifurcation point and, with a slight abuse of language, we will say that t_0 is a *regular value* of v_{nA} if it is neither a critical nor a bifurcation value.

By the aforementioned lemma of Taubes, if t is a regular value of v_{nA} and $C \in \mathcal{M}_{mA}(X, J_t)$ is an embedded torus for some $m \leq n$, the signs of C and its relevant covers are defined, and so we can compute the functions $P(C, \cdot)$ and $P_\pi(C)$.

Proposition 3.16. *Let (W, X, π, F) be a surface bundle and $\{\omega_t \mid t \in [0, 1]\}$ a smooth path of symplectic structures on X . Let $\{J_t \mid t \in [0, 1]\}$ be a path of ω_t -compatible*

almost complex structures given by Lemma 2.4. Then for any $A \in \mathcal{T}_2(X)$ primitive, if t is a regular value of v_{mA} for all m , the product

$$\prod_{m>0} \left(\prod_{C \in \mathcal{M}_{mA}(X, J_t)} P_\pi(C) \right)$$

does not depend on the choice of t .

Proof. Let t_0 be either a critical or a bifurcation value of the map v_{nA} for some $n \in \mathbb{N}$ and let $r > 0$ such that t_0 is the only non-regular value of v_{nA} in the interval $(t_0 - r, t_0 + r)$. We want to prove that for any $0 < s < r$

$$(3.10) \quad \prod_{m \leq n} \left(\prod_{C \in \mathcal{M}_{mA}(X, J_{t_0-s})} P_\pi(C) \right) = \prod_{m \leq n} \left(\prod_{C \in \mathcal{M}_{mA}(X, J_{t_0+s})} P_\pi(C) \right).$$

Note first that, since $(t_0 - r, t_0) \cup (t_0, t_0 + r)$ contains only regular values of v_{nA} , the two sides of (3.10) do not depend on the choice of s .

Let us first check the case where t_0 corresponds to a critical point. As in the proof of Corollary 3.15, t_0 corresponds to the birth or a death of a pair of isotopic embedded tori C and C' both in $v_{nA}^{-1}(t_0 + s)$ or, respectively, $v_{nA}^{-1}(t_0 - s)$. By [22, Lemma 5.9]

$$\epsilon(C) = -\epsilon(C') \quad \text{and} \quad \epsilon(C_\iota) = \epsilon(C'_\iota)$$

where ι is any of the connected covers $\{\iota_1, \iota_2, \iota_3, \iota_4\}$, so that if C is of type (ϵ, I) then C' is of type $(-\epsilon, I)$. Moreover, since C and C' are isotopic, for any ι we have $\text{GT}_X(\pi^* C_\iota) = \text{GT}_X(\pi^* C'_\iota)$. Summarizing:

$$P_\pi(C') = (P_\pi(C))^{-1}$$

and the total contribution of C and C' to both sides of (3.10) is

$$P_\pi(C)P_\pi(C') = 1.$$

Let us prove now the equivalence in (3.10) for t_0 corresponding to a bifurcation point. This means that there exists $B = kA$ for some k and a sequence of tori in $\mathcal{C}_{2B}(X, \{J_t\})$ that converges to a double cover $\bar{\iota}$ of a torus $C_0 \in \mathcal{M}_B(X, J_{t_0}) \subset \mathcal{C}_B(X, \{J_t\})$. There are two possibilities, depending on whether the sequence converges to C_0 for increasing or decreasing t . We will assume t decreasing and we will leave the (completely analogue) proof for the other case to the reader.

The connected component of $\mathcal{C}_B(X, \{J_t\})$ containing C_0 gives an isotopy between the two embedded tori $C_- \subset \mathcal{M}_B(X, J_{t_0-s})$ and $C_+ \subset \mathcal{M}_B(X, J_{t_0+s})$, so that

$$(3.11) \quad \text{GT}_X(\pi^* C_-) = \text{GT}_X(\pi^* C_0) = \text{GT}_X(\pi^* C_+)$$

and, for any $\iota \in \{\iota_1, \iota_2, \iota_3, \iota_4\}$,

$$(3.12) \quad \text{GT}_X(\pi^*(C_-)_\iota) = \text{GT}_X(\pi^*(C_0)_\iota) = \text{GT}_X(\pi^*(C_+)_\iota),$$

where, as usual, we identified the covers of different tori using the isotopy. By Lemma 5.10 of [22] we can recover the relative signs of C_- , C_+ and their double covers:

$$(3.13) \quad \begin{aligned} \epsilon(C_-) &= \epsilon(C_+); \\ \epsilon((C_-)_{\bar{\iota}}) &= -\epsilon((C_+)_{\bar{\iota}}); \\ \epsilon((C_-)_\iota) &= \epsilon((C_+)_\iota) \text{ for any double cover } \iota \neq \bar{\iota}. \end{aligned}$$

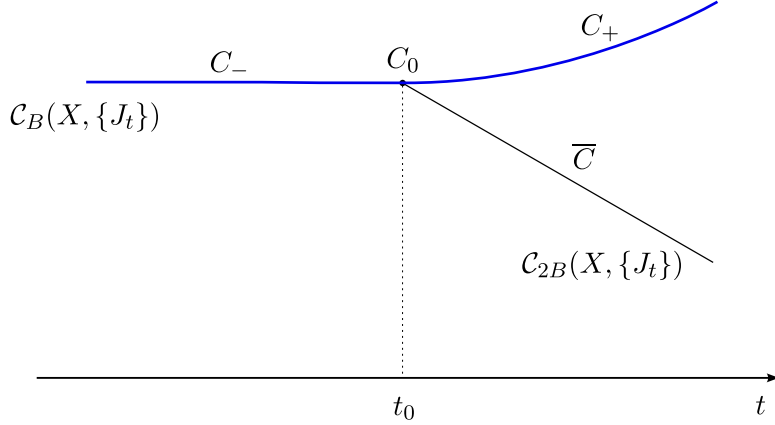


FIGURE 1. Bifurcation.

Now, let \overline{C} be the only torus in $\mathcal{M}_{2B}(X, J_{t_0+s})$ belonging to the relevant connected component of $\mathcal{C}_{2B}(X, \{J_t\})$. This connected component gives a homotopy between \overline{C} and the double cover $(C_0)_{\overline{\iota}}$ (and so also with the double covers $(C_-)_{\overline{\iota}}$ and $(C_+)_{\overline{\iota}}$). Then, by homotopy invariance and (3.12), we have

$$(3.14) \quad \text{GT}_X(\pi^*(\overline{C})) = \text{GT}_X(\pi^*((C_-)_{\overline{\iota}})) = \text{GT}_X(\pi^*((C_+)_{\overline{\iota}})).$$

Moreover, observe that the 4-fold cover ι_4 of C_0 given by Lemma 3.6 naturally induces a double cover $\widehat{\iota}$ of \overline{C} . More in detail, as for $s \rightarrow 0$ \overline{C} converges to (the total space of) the double cover $\overline{\iota}$ of C_0 , we require that, for $s \rightarrow 0$, $\overline{C}_{\widehat{\iota}}$ converges to (the total space of) the double cover $f_{\overline{\iota}}^*(\iota)$ of $(C_0)_{\overline{\iota}}$ for any (non-trivial) $\iota \neq \overline{\iota}$. Since $f_{\overline{\iota}}^*(\iota)$ is exactly the cover ι_4 of C_0 , again by isotopy invariance, we have:

$$(3.15) \quad \text{GT}_X(\pi^*(\overline{C}_{\widehat{\iota}})) = \text{GT}_X(\pi^*((C_-)_{\iota_4})) = \text{GT}_X(\pi^*((C_+)_{\iota_4})).$$

By Lemma 5.11 of [22] we can recover the signs of \overline{C} and of its double covers:

$$(3.16) \quad \begin{aligned} \epsilon(\overline{C}) &= -\epsilon(C_+)\epsilon((C_+)_{\overline{\iota}}); \\ \epsilon(\overline{C}_{\widehat{\iota}}) &= \prod_{\iota \neq \widehat{\iota}} \epsilon((C_+)_{\iota}); \\ \epsilon(\overline{C}_{\iota}) &= +1 \text{ for } \iota \neq \widehat{\iota}. \end{aligned}$$

We are now ready to prove that the equivalence (3.10) holds also when we cross a bifurcation. To prove the result it is enough to show that

$$P_{\pi}(C_+) \cdot P_{\pi}(\overline{C}) = P_{\pi}(C_-).$$

We will check only the three cases below, corresponding to the three possible types $(+, I)$ of C_- with $\epsilon(C_-) = +$ and $\overline{\iota} \notin I$: if $\epsilon(C_-) = -$ each case will be the reciprocal of the corresponding case for $\epsilon(C_-) = +$, while the cases with $\overline{\iota} \in I$ do not give new relations and are left to the reader.

- $[I = \emptyset]$: C_+ is of type $(+, \{\bar{\iota}\})$ and \bar{C} is of type $(+, \emptyset)$. Then:

$$\begin{aligned} P_\pi(C_+) \cdot P_\pi(\bar{C}) &= \frac{\text{GT}_X(\pi^*C_+)}{\text{GT}_X(\pi^*((C_+)_{\bar{\iota}}))} \cdot \text{GT}_X(\pi^*\bar{C}) \\ &= \text{GT}_X(\pi^*C_+) \\ &= \text{GT}_X(\pi^*C_-) = P_\pi(C_-). \end{aligned}$$

- $[I = \{\iota'\}]$: C_+ is of type $(+, \{\bar{\iota}, \iota'\})$ and \bar{C} is of type $(+, \{\hat{\iota}\})$, where $\hat{\iota}$ is as above. Then:

$$\begin{aligned} P_\pi(C_+) \cdot P_\pi(\bar{C}) &= \frac{\text{GT}_X(\pi^*C_+) \text{GT}_X(\pi^*((C_+)_{\iota_4}))}{\text{GT}_X(\pi^*((C_+)_{\bar{\iota}})) \text{GT}_X(\pi^*((C_+)_{\iota'}))} \cdot \frac{\text{GT}_X(\pi^*\bar{C})}{\text{GT}_X(\pi^*\bar{C}_{\hat{\iota}})} \\ &= \frac{\text{GT}_X(\pi^*C_+)}{\text{GT}_X(\pi^*((C_+)_{\iota'}))} \\ &= \frac{\text{GT}_X(\pi^*C_-)}{\text{GT}_X(\pi^*((C_-)_{\iota'}))} = P_\pi(C_-). \end{aligned}$$

- $[I = \{\iota', \iota''\}]$: C_+ is of type $(+, \{\bar{\iota}, \iota', \iota''\})$ and \bar{C} is of type $(+, \emptyset)$ (since by (3.16) we have $\epsilon(\bar{C}_{\hat{\iota}}) = \epsilon((C_+)_{\iota'}) \cdot \epsilon((C_+)_{\iota''}) = (-1) \cdot (-1) = +1$). Then:

$$\begin{aligned} P_\pi(C_+) \cdot P_\pi(\bar{C}) &= \frac{\text{GT}_X(\pi^*C_+) \text{GT}_X(\pi^*((C_+)_{\iota_4})) \cdot \text{GT}_X(\pi^*\bar{C})}{\text{GT}_X(\pi^*((C_+)_{\bar{\iota}})) \text{GT}_X(\pi^*((C_+)_{\iota'})) \text{GT}_X(\pi^*((C_+)_{\iota''}))} \\ &= \frac{\text{GT}_X(\pi^*C_+) \text{GT}_X(\pi^*((C_+)_{\iota_4}))}{\text{GT}_X(\pi^*((C_+)_{\iota'})) \text{GT}_X(\pi^*((C_+)_{\iota''}))} \\ &= \frac{\text{GT}_X(\pi^*C_-) \text{GT}_X(\pi^*((C_-)_{\iota_4}))}{\text{GT}_X(\pi^*((C_-)_{\iota'})) \text{GT}_X(\pi^*((C_-)_{\iota''}))} = P_\pi(C_-). \end{aligned}$$

□

Corollary 3.17. *Let (W, X, π, F) be a surface bundle, $A \in \mathcal{T}_2(X)$ and $\{\omega_t \mid t \in [0, 1]\}$ a smooth path of symplectic structures on X . Then*

$$\mathfrak{Qu}_\pi(A, m, \omega_0) = \mathfrak{Qu}_\pi(A, m, \omega_1).$$

Theorem 1.1 follows then from the definition of $\text{GT}_\pi(X, \omega, J)$ and corollaries 3.14, 3.15 and 3.17.

4. MAPPING TORI AND TWISTED LESFCHETZ ZETA FUNCTIONS

In this section we first review the definition of standard and twisted Lefschetz zeta functions of surface diffeomorphisms. We then define the “bundled twistings” for Lefschetz zeta functions and prove the equivalence with certain Jiang’s twisted Lefschetz zeta functions. Finally we compute the twisted Gromov-Taubes invariants for symplectic closed 4-manifolds of the form $S^1 \times Y$ and give a proof of Theorem 1.2 of introduction.

4.1. Commutative Lefschetz zeta functions.

Let S be an oriented compact connected surface and let $\phi : S \rightarrow S$ be an orientation preserving diffeomorphism. A point $x \in S$ is a *fixed point* of ϕ if $\phi(x) = x$. More in general, given $n \in \mathbb{N}^*$, x is a *periodic point of period n* (or *n -periodic point*) of ϕ if $\phi^n(x) = x$. An n -periodic point x is said to be *non-degenerate* if

$$\det(\mathbb{1} - d_x \phi^n) \neq 0.$$

The diffeomorphism ϕ is *non-degenerate* if for every $n \in \mathbb{N}^*$ there is only a finite number of n -periodic points, each of which is non-degenerate. Note in particular that in this case an n -periodic point x is, for every $p \in \mathbb{N}^*$, also a pn -periodic point, which is non-degenerate with respect to ϕ^{pn} . It is well known that every element of the oriented mapping class group $\text{MCG}(S)$ has a representative that is smooth and non degenerate. From now on we will then assume the non-degeneracy of ϕ .

The *Lefschetz sign* of an n -periodic point x is

$$\varepsilon(x, n) := \text{sign}(\det(\mathbb{1} - d_x \phi^n)).$$

Observe that any n -periodic point of ϕ can be naturally interpreted as a fixed point of ϕ^n and that the Lefschetz signs given by the two interpretations are the same. On the other hand, we remark that in general $(\varepsilon(x, n))^p \neq \varepsilon(x, pn)$ for $p > 1$.

Let now

$$T_\phi := \frac{S \times [0, +\infty)}{(x, t+1) \sim (\phi(x), t)}$$

be the *mapping torus* of (S, ϕ) . We will often use the identification $S = S \times \{0\} \subset T_\phi$.

Let t be the coordinate of $[0, +\infty)$ in T_ϕ . An *n -periodic orbit* of T_ϕ is a closed positively oriented orbit δ of the flow on T_ϕ of the vector field ∂_t , such that $\langle \delta, S \rangle = n$ and considered up to reparametrization. An orbit is *simple* if it is not a non-trivial cover of another closed orbit of the flow. Any orbit can then be identified with some p -fold cover δ^p of some simple orbit δ .

To any n -periodic simple orbit δ corresponds in a natural way the set $\{\delta \cap S\}$ of n periodic points, each of period n : the *Lefschetz sign* of δ is defined by

$$\varepsilon(\delta) := \varepsilon(x, n) \text{ for any } x \in \{\delta \cap S\}.$$

In general, given an integer $p > 0$, the Lefschetz sign of δ^p is

$$\varepsilon(\delta^p) := \varepsilon(x, pn) \text{ for any } x \in \{\delta \cap S\}.$$

Definition 4.1. Given a simple orbit δ of (S, ϕ) , the *local Lefschetz zeta function* of δ is the formal power series

$$\zeta_\delta(t) := \exp \left(\sum_{m \geq 1} \varepsilon(\delta^m) \frac{t^m}{m} \right) \in \mathbb{Z}[[t]].$$

Remark 4.2. For the following, we remark that it is possible to give more explicit formulas for $\zeta_\delta(t)$ depending on the *type* of δ . It is a standard result that there are essentially two types of non-degenerate periodic orbits: *elliptic* and *hyperbolic*. A simple n -periodic orbit δ is elliptic if, for any $x \in \delta \cap S$, the eigenvalues of $d_x \phi^n$ are complex conjugates and is hyperbolic if they are real. In the last case we can make a further distinction: δ is called *positive* (resp. *negative*) hyperbolic if the eigenvalues of $d_x \phi^n$ are both positive (resp. negative). It is then easy to prove that:

$$(4.1) \quad \varepsilon(\delta^m) = \begin{cases} +1 & \text{if } \delta \text{ elliptic;} \\ -1 & \text{if } \delta \text{ positive hyperbolic;} \\ (-1)^{m+1} & \text{if } \delta \text{ negative hyperbolic} \end{cases}$$

and the following expressions follow (see for example Lemma 3.15 of [21]):

$$(4.2) \quad \zeta_\delta(t) = \begin{cases} \frac{1}{1-t} = 1 + t + t^2 + \dots & \text{if } \delta \text{ elliptic;} \\ 1 - t & \text{if } \delta \text{ positive hyperbolic;} \\ 1 + t & \text{if } \delta \text{ negative hyperbolic.} \end{cases}$$

Definition 4.3. The *Lefschetz zeta function* of ϕ is

$$\zeta(\phi) := \prod_{\delta \text{ simple}} \zeta_\delta \left(t^{(\delta, S)} \right) \in \mathbb{Z}[[t]]$$

where the product is meant to be over all the simple periodic orbits δ of ϕ .

We observe that in $\zeta(\phi)$ any periodic orbit of ϕ is counted exactly once and with weight depending only on its sign and its homology class in T_ϕ . The Lefschetz zeta function can be expressed also in terms of the *Lefschetz numbers* of the iterates of ϕ .

Definition 4.4. Given (S, ϕ) , for $i = 0, 1, 2$ let $\phi_i : H_i(S) \rightarrow H_i(S)$ be the induced isomorphisms in homology. The *Lefschetz number* of ϕ is:

$$L(\phi) := \sum_{i=0}^2 (-1)^i \text{tr}(\phi_i).$$

Lefschetz fixed point theorem. Let $\text{Fix}(\phi)$ denote the set of fixed points of ϕ . Then:

$$L(\phi) = \sum_{x \in \text{Fix}(\phi)} \varepsilon(x, 1).$$

Corollary 4.5. $\zeta(\phi) = \exp \left(\sum_{n \geq 1} L(\phi^n) \frac{t^n}{n} \right).$

The proof of last corollary is matter of express any n -periodic orbit of ϕ as a fixed point of ϕ^n , reorganize the product in the definition of $\zeta(\phi)$ as a product in $n \geq 1$ and then apply the Lefschetz fixed point theorem (see for example [5]).

The last corollary implies in particular that $\zeta(\phi)$ is a topological invariant of T_ϕ . Moreover we have the following corollary (see for example [1]).

Corollary 4.6. Let $\tau(T_\phi) = \tau(T_\phi, t)$ denote the Reidemeister torsion of T_ϕ . Then:

$$\zeta(\phi) = \prod_{i=0}^2 (\det(\mathbb{1} - t\phi_i))^{(-1)^{i+1}} \doteq \tau(T_\phi)$$

where \doteq denotes the equivalence up to multiplications by monomials of the form $\pm t^m$.

If not stated otherwise, all the Reidemeister torsions of mapping tori will be considered with the normalization given by the co-oriented 0-page S , so that the equivalence \doteq with the corresponding zeta functions will be in fact an equality.

There are various refinements of $\zeta(\phi)$: the richest Abelian that one can define is obtained by twisting the contribution of each orbit by its total homology class in T_ϕ . By “Abelian” we mean that it can be obtained by using an Abelian representation of $\pi_1(T_\phi)$. This refinement is obtained by encoding the elements $a \in H_1(T_\phi)$ in formal variables t_a such that $t_a \cdot t_b = t_{a+b}$ like those in the definition of GT and then considering the *total Abelian Lefschetz zeta function*:

$$(4.3) \quad \zeta_A(\phi) := \prod_{\delta \text{ simple}} \zeta_\delta \left(t_{[\delta]} \cdot t^{(\delta, S)} \right) \in \mathbb{Z}[[H_1(T_\phi)]] [[t]].$$

The series $\zeta_A(\phi)$ should be thought of as a series in the formal variable t and coefficients in $\mathbb{Z}[[H_1(T_\phi)]]$.

All the results above about $\zeta(\phi)$ can be generalized to $\zeta_A(\phi)$ by introducing corresponding twisting coefficients for the Lefschetz numbers and the Reidemeister torsion (see [5], [6] or [10]). Briefly, instead of considering the homomorphisms induced by ϕ in homology, one considers the homeomorphism

$$\tilde{\phi} : \tilde{S} \longrightarrow \tilde{S}$$

induced by ϕ on the universal Abelian cover \tilde{S} of S . Given a cellular decomposition of \tilde{S} obtained by lifting a cellular decomposition of S , we can assume that both ϕ and $\tilde{\phi}$ are cellular maps. Then, choosing basis and taking local $\mathbb{Z}[H_1(S)]$ -coefficients for $H_i(\tilde{S})$, $i = 0, 1, 2$, $\tilde{\phi}$ gives rise to non-singular $\mathbb{Z}[\text{coker}(\mathbb{1} - \phi_1)]$ -matrices $\tilde{\phi}_i$ of order $\text{rk}(H_i(S))$. The Mayer-Vietoris exact sequence implies that

$$(4.4) \quad H_1(T_\phi) \cong \text{coker}(\mathbb{1} - \phi_1) \oplus \mathbb{Z}_\mu$$

so that the matrices $\tilde{\phi}_i$ can be interpreted as matrices with coefficients in $\mathbb{Z}[H_1(T_\phi)]$. The *total Abelian Lefschetz number* of ϕ is

$$(4.5) \quad L_A(\phi) := \sum_{i=0}^2 (-1)^i \text{tr}(t_\mu \tilde{\phi}_i) \in \mathbb{Z}[H_1(T_\phi)].$$

where t_μ is a formal variable encoding the homology class of the generator μ of $H_1(T_\phi)$. The *total Abelian Lefschetz fixed point Theorem* (cf. Theorem 1 of [6]) gives then

$$(4.6) \quad L_A(\phi) = \sum_{x \in \text{Fix}(\phi)} \varepsilon(x, 1) [\delta_x] \in \mathbb{Z}[H_1(T_\phi)]$$

where δ_x is the unique 1-periodic orbit of ϕ with $\delta \cap S = \{x\}$. We have then the following analogue of Corollary 4.5:

$$(4.7) \quad \zeta_A(\phi) = \exp \left(\sum_{n \geq 1} L_A(\phi^n) \frac{t^n}{n} \right) \in \mathbb{Z}[[H_1(T_\phi)]].$$

Similarly, the generalization to the total Abelian case of Corollary 4.6 gives:

$$(4.8) \quad \zeta_A(\phi) = \prod_{i=0}^2 \left(\det \left(\mathbb{1} - t(t_\mu \tilde{\phi}_i) \right) \right)^{(-1)^{i+1}} = \tau_A(T_\phi, t).$$

where $\tau_A(T_\phi, t)$ is the *total Abelian Reidemeister torsion* of T_ϕ .

Example 4.7. Let $L = K_1 \sqcup \dots \sqcup K_n$ be a fibered n -component link in a homology 3-sphere Y and let (S, ϕ) be the corresponding open book decomposition of Y (so that in particular $T_\phi \cong Y \setminus L$). Let t_i , $i \in \{1, \dots, n\}$ be formal variables encoding the homology classes of the meridians of the K_i 's. Then we can express all the variables t_a , $a \in H_1(T_\phi)$ in the definition of $\zeta_A(\phi)$ in terms of the t_i 's and:

$$(4.9) \quad \zeta_A(\phi)|_{t=1} = \tau_A(Y \setminus L) = \begin{cases} \frac{\Delta_L(t_1)}{1 - t_1} & \text{if } n = 1 \\ \Delta_L(t_1, \dots, t_n) & \text{if } n > 1 \end{cases}$$

where $\zeta_A(\phi)|_{t=1}$ is the evaluation in $t = 1$ of the series $\zeta_A(\phi)$ and Δ_L denotes the *multivariable Alexander polynomial* of L .

4.2. Non-commutative Lefschetz zeta functions.

In this subsection we recall the *twisted Lefschetz zeta functions* defined by Jiang in [16] (see [17]). Let (S, ϕ) be as before, let R be an Abelian ring with unity and $n \in \mathbb{N}^*$. Choose a base point $y_0 \in S \subset T_\phi$ and consider a representation

$$\rho : \pi_1(T_\phi) \longrightarrow \mathrm{GL}(n, R)$$

of the fundamental group $\pi_1(T_\phi)$ (for simplicity in the notation we will avoid to refer to y_0) into the linear group over R of order n . Then ρ extends to a representation (still denoted ρ)

$$\rho : \mathbb{Z}[\pi_1(T_\phi)] \longrightarrow \mathrm{M}(n, R)$$

of the group ring $\mathbb{Z}[\pi_1(T_\phi)]$ into the algebra of $n \times n$ matrices over R .

For any simple orbit δ of T_ϕ , choose a path γ_δ from y_0 to any point of δ . Then, for any $m \geq 0$, $\gamma_\delta * \delta^m * \gamma_\delta^{-1}$ can be regarded as a closed path based on y_0 .

Definition 4.8. Given a simple orbit δ of (S, ϕ) , the *local ρ -twisted Lefschetz zeta function* of δ is the formal power series

$$\zeta_\delta(\rho, t) := \exp \left(\sum_{m \geq 1} \varepsilon(\delta^m) \mathrm{tr}(\rho([\gamma_\delta * \delta^m * \gamma_\delta^{-1}])) \frac{t^m}{m} \right) \in R[[t]].$$

Observe that $\zeta_\delta(\rho, t)$ does not depend on the choice of the path γ_δ (since the trace is invariant under conjugation), and so it depends only on the free homotopy class of δ . By a slight abuse of notation we will omit the path γ_δ in the notation and we will write just

$$\zeta_\delta(\rho, t) = \exp \left(\sum_{m \geq 1} \varepsilon(\delta^m) \mathrm{tr}(\rho([\delta^m])) \frac{t^m}{m} \right).$$

Definition 4.9. The *ρ -twisted Lefschetz zeta function* of ϕ is

$$\zeta_\rho(\phi) := \prod_{\delta \text{ simple}} \zeta_\delta(\rho, t^{(\delta, S)}) \in R[[t]].$$

The ρ -twisted Lefschetz zeta functions enjoy similar properties to those of the commutative ones. In particular a result analogue to Corollary 4.6 holds:

$$(4.10) \quad \zeta_\rho(\phi) = \tau_\rho(T_\phi)$$

where $\tau_\rho(T_\phi) = \tau_\rho(T_\phi, t)$ is the Lin's ρ -twisted Reidemeister torsion of T_ϕ (see [17] and [18] for the details). In particular $\zeta_\rho(\phi)$ depends only on the conjugacy class of ρ and the isotopy class of ϕ . The set $\{\zeta_\rho(\phi)\}_\rho$ is then a topological invariant of T_ϕ .

4.3. Twisted Lefschetz zeta functions associated with bundles.

Let F be an oriented compact connected surface (possibly with boundary) and

$$F \hookrightarrow V \xrightarrow{\pi} T_\phi$$

be a smooth oriented fiber bundle. Let

$$(4.11) \quad \mathbf{m} = \mathbf{m}_\pi : \pi_1(T_\phi) \longrightarrow \mathrm{MCG}(F)$$

be the *monodromy homomorphism* of the bundle into the oriented mapping class group of F . If δ is an orbit of ϕ and γ_δ is a path from the base point for $\pi_1(T_\phi)$ to

δ like in last subsection, then $\mathbf{m}([\gamma_\delta \delta \gamma_\delta^{-1}])$ depends only on the free homotopy class of δ , so that $\mathbf{m}([\delta])$ is well defined. For any simple orbit δ of ϕ , let

$$\psi_\delta : F \longrightarrow F$$

be a smooth non-degenerate representative of $\mathbf{m}([\delta])$ and, for any $n > 1$, set $\psi_{\delta^n} = \psi_\delta^n$. Now, the total space $\pi^*\delta$ of the restriction of π to δ naturally fibers over $\delta \cong S^1$ with fiber F and we have a diffeomorphism

$$\pi^*\delta \cong T_{\psi_\delta}.$$

Remark 4.10. Given any simple orbit δ of T_ϕ , by (4.4) it follows that

$$H_1(\pi^*\delta) \cong \text{coker}(\mathbb{1} - \psi_{\delta_1}) \oplus \mathbb{Z}_l$$

where the generator l can be chosen in a way that

$$\pi_* \circ i_*(l) = [\delta] \in H_1(T_\phi)$$

where $i : \pi^*\delta \hookrightarrow V$ is the natural inclusion. Moreover

$$\ker(\pi_* \circ i_*|_{H_1(T_{\psi_\delta})}) = \text{coker}(\mathbb{1} - \psi_{\delta_1}).$$

In particular for any orbit γ in T_{ψ_δ} :

$$\pi_* \circ i_*([\gamma]) = \langle \gamma, F \rangle [\delta].$$

where $\langle \gamma, F \rangle$ denotes the algebraic intersection number in T_{ψ_δ} between γ and the surface $F = F \times \{0\} \subset T_{\psi_\delta}$. Then, for the usual formal variables t_a encoding classes $a \in H_1(T_\phi)$, there is a natural identification

$$t_{\pi_* \circ i_*([\gamma])} = t_{\langle \gamma, F \rangle [\delta]} = t_{[\delta]}^{\langle \gamma, F \rangle}.$$

Notation 4.11. Similarly to Notation 3.3, we will often regard $\pi^*\delta$ as a submanifold of V and if $b \in H_1(\pi^*\delta)$, $b_V \in H_1(V)$ will denote the image of b under the homomorphism induced by the inclusion of $\pi^*\delta$ in V . Moreover we will often assume the identification $\pi^*\delta = T_{\psi_\delta}$ without mentioning it.

Definition 4.12. Let (V, T_ϕ, π, F) be a surface bundle as above and let δ be a simple periodic orbit of ϕ . We define the *local π -twisted Lefschetz zeta function*

$$\zeta_\delta(\pi, t) \in \mathbb{Z}[[H_1(V)]][[t]]$$

of δ by:

$$\zeta_\delta(\pi, t) := \prod_{\substack{\gamma \text{ simple} \\ \text{in } T_{\psi_\delta}}} \exp \left(\sum_{m \geq 1} \varepsilon(\delta^{\langle \gamma, F \rangle} \cdot m) \varepsilon(\gamma^m) \frac{(z_{[\gamma]_V} \cdot t^{\langle \gamma, F \rangle})^m}{m} \right).$$

where the variables z_d keep track of homology classes $d \in H_1(V)$.

Definition 4.13. Given (V, T_ϕ, π, F) , we define the *π -twisted Lefschetz zeta function*

$$\zeta_\pi(\phi) \in \mathbb{Z}[[H_1(V) \oplus H_1(T_\phi)]][[t]]$$

of ϕ by:

$$\zeta_\pi(\phi) := \prod_{\substack{\delta \text{ simple} \\ \text{in } T_\phi}} \zeta_\delta \left(\pi, t_{[\delta]} \cdot t^{\langle \delta, S \rangle} \right).$$

where the variables t_a keep track of homology classes $a \in H_1(T_\phi)$.

4.4. Relation with the non-commutative Lefschetz zeta functions.

Given a surface bundle (V, T_ϕ, π, F) , we want in some sense the “richest representation” of $\pi_1(T_\phi)$ induced by \mathbf{m}_π into a linear group over some Abelian ring. In what follows we will define this in the fibered case, but the construction can be carried on in a completely analogous way for a general 3-manifold (cf. Definition 4.19 below).

Given (V, T_ϕ, π, F) , the associated monodromy map $\mathbf{m} = \mathbf{m}_\pi$ in (4.11) induces in homology an *algebraic monodromy of index $i \in \{0, 1, 2\}$*

$$(4.12) \quad \begin{array}{ccc} \mathbf{m}_i = \mathbf{m}_i^\pi & : & \pi_1(T_\phi) \longrightarrow \mathrm{GL}(H_i(F)) \\ & & \alpha \longmapsto \mathbf{m}_i(\alpha) \end{array}$$

where $r_i = \mathrm{rank}(H_i(F))$ and $\mathbf{m}_i(\alpha)$ is the matrix associated to the homomorphism induced in homology by (any representative of) $\mathbf{m}(\alpha)$. In order to get all the Abelian information that we can, we proceed as in Subsection 4.1. Given a class $\alpha \in \pi_1(T_\phi)$, we have

$$\pi^* \alpha \stackrel{\text{homeo}}{\cong} T_{\mathbf{m}(\alpha)}$$

where, with a slight abuse of notation, we keep to use α instead of taking a representative. Let l_α denote the generator of

$$H_1(\pi^* \alpha) \cong \mathrm{coker}(\mathbb{1} - \mathbf{m}_1(\alpha)) \oplus \mathbb{Z}_{l_\alpha}$$

as in Remark 4.10 and satisfying in particular the relation $\pi_* \circ i_*(l_\alpha) = [\alpha] \in H_1(T_\phi)$.

Then, as we have seen recalling the definition of the total Abelian Lefschetz numbers, for $i \in \{0, 1, 2\}$, we can go to the universal Abelian cover of F and consider the *total Abelian algebraic monodromies*

$$\begin{array}{ccc} \tilde{\mathbf{m}}_i & : & \pi_1(T_\phi) \longrightarrow \mathrm{GL}(r_i, \mathbb{Z}[\mathrm{coker}(\mathbb{1} - \mathbf{m}_1(\alpha))]) \\ & & \alpha \longmapsto \tilde{\mathbf{m}}_i(\alpha) \end{array}$$

These induce representations

$$\begin{array}{ccc} \tilde{\mathbf{m}}_i^V & : & \pi_1(T_\phi) \longrightarrow \mathrm{GL}(r_i, \mathbb{Z}[H_1(V)]) \\ & & \alpha \longmapsto z_{(l_{\alpha V})} \cdot i_*(\tilde{\mathbf{m}}_i(\alpha)) \end{array}$$

where, as usual, i_* denotes the map induced by the natural inclusion $i : \pi^* \alpha \hookrightarrow V$ and we associate variables z_{c_V} (encoding classes in $H_1(V)$) to classes $c \in H_1(\pi^* \alpha)$. Consider now the ring homomorphism

$$\begin{array}{ccc} p : \mathbb{Z}[H_1(V)] & \longrightarrow & \mathbb{Z}[H_1(V) \oplus H_1(T_\phi)] \\ az_b & \longmapsto & az_b \cdot t_{\pi_* b} \end{array}$$

and the induced representations

$$p : \mathrm{GL}(r_i, \mathbb{Z}[H_1(V)]) \longrightarrow \mathrm{GL}(r_i, \mathbb{Z}[H_1(V) \oplus H_1(T_\phi)]).$$

Definition 4.14. Given the surface bundle (V, T_ϕ, π, F) , the *total Abelian monodromy representation of $\pi_1(T_\phi)$ of index $i \in \{0, 1, 2\}$ induced by π* is

$$\begin{array}{ccc} \rho_i^\pi : \pi_1(T_\phi) & \longrightarrow & \mathrm{GL}(r_i, \mathbb{Z}[H_1(V) \oplus H_1(T_\phi)]) \\ & & \alpha \longmapsto p(\tilde{\mathbf{m}}_i^V(\alpha)) \end{array}$$

Theorem 4.15. *Given a surface bundle (V, T_ϕ, π, F) with induced total Abelian monodromy representations ρ_i^π we have*

$$\zeta_\pi(\phi) = \prod_{i=0}^2 (\zeta_{\rho_i^\pi}(\phi))^{(-1)^i}$$

where $\zeta_{\rho_i^\pi}(\phi)$ is the ρ_i^π -twisted Lefschetz zeta function of ϕ .

Corollary 4.16. *The π -twisted Lefschetz zeta function of ϕ depends only on the homeomorphism class of T_ϕ and the bundle isomorphism class of (V, T_ϕ, π, F) . In particular $\zeta_\pi(\phi)$ does not depend on the choice of the representatives ψ_δ of $\mathbf{m}([\delta])$.*

Proof of Theorem 4.15. Before going into the computations we give the basic idea behind the theorem. If δ is a simple orbit of ϕ , then the local π -twisted Lefschetz zeta function $\zeta_\delta(\pi, t)$ of δ can be interpreted as a kind of “global” total Abelian Lefschetz function of the manifold T_{ψ_δ} for a fixed representative ψ_δ of $\mathbf{m}([\delta])$. Using the definition in (4.5) and the relations (4.6) and (4.7) it is possible to express the count of orbits in $\zeta_\delta(\pi, t)$ in terms of traces of powers of the matrices $\tilde{\mathbf{m}}_i([\delta])$.

The actual computation requires a bit of attention because we want to keep track of the homology classes of the orbits in T_{ψ_δ} and also of their projections in $H_1(T_\phi)$, which are counted via the more accurate representations ρ_i^π .

Let us begin by translating the orbit count in $\zeta_\delta(\pi, t^{\langle \delta, S \rangle})$ in terms of fixed points of the iterates of the representative ψ_δ of $\mathbf{m}([\delta])$.

$$\begin{aligned} \zeta_\delta(\pi, t_{[\delta]} \cdot t^{\langle \delta, S \rangle}) &= \\ &= \prod_{\substack{\gamma \text{ simple} \\ \text{in } T_{\psi_\delta}}} \exp \left(\sum_{m \geq 1} \varepsilon(\delta^{\langle \gamma, F \rangle \cdot m}) \varepsilon(\gamma^m) \frac{(z_{[\gamma]}_V)^m \cdot (t_{[\delta]} \cdot t^{\langle \delta, S \rangle})^{\langle \gamma, F \rangle \cdot m}}{m} \right) \\ &= \exp \left(\sum_{m \geq 1} \sum_{\substack{\gamma \text{ simple} \\ \text{in } T_{\psi_\delta}}} \varepsilon(\delta^{\langle \gamma, F \rangle \cdot m}) \varepsilon(\gamma^m) z_{[\gamma^m]}_V t_{\pi_*([\gamma^m]_V)} \frac{(t^{\langle \delta, S \rangle})^{\langle \gamma, F \rangle \cdot m}}{m} \right) \\ &= \exp \left(\sum_{n \geq 1} \sum_{x \in \text{Fix}(\psi_\delta^n)} \varepsilon(\delta^n) \varepsilon(x, n) (z_{[\gamma_x]}_V t_{\pi_*([\gamma_x]_V)}) \frac{t^{\langle \delta, S \rangle \cdot n}}{n} \right). \end{aligned}$$

Here $n = \langle \gamma, F \rangle \cdot m$ and γ_x is the unique n -periodic orbit in T_{ψ_δ} passing through x and has homology class $[\gamma_x] \in H_1(T_{\psi_\delta})$. By (4.7) and the definition of L_A in (4.5) we get:

$$\begin{aligned} \zeta_\delta(\pi, t_{[\delta]} \cdot t^{\langle \delta, S \rangle}) &= \exp \left(\sum_{n \geq 1} \varepsilon(\delta^n) \sum_{i=0}^2 (-1)^i \text{tr}(\rho_i^\pi([\delta^n])) \frac{t^{\langle \delta, S \rangle \cdot n}}{n} \right) \\ &= \prod_{i=0}^2 \left(\exp \left(\sum_{n \geq 1} \varepsilon(\delta^n) \text{tr}(\rho_i^\pi([\delta^n])) \frac{t^{\langle \delta, S \rangle \cdot n}}{n} \right) \right)^{(-1)^i} \\ &= \prod_{i=0}^2 (\zeta_\delta(\rho_i^\pi, t))^{(-1)^i}. \end{aligned}$$

The result follows then by multiplying over all the simple periodic orbits δ of ϕ . \square

In the proof of last theorem we regarded the contribution to $\zeta_\pi(\phi)$ of each simple periodic orbit δ of ϕ as a kind of total Abelian Lefschetz zeta function of the manifold $\pi^*\delta$, evaluated in the image in $H_1(V) \oplus H_1(T_\phi)$ of $H_1(\pi^*\delta)$ via the homomorphism $i_* \oplus (\pi \circ i)_*$, where $i : \pi^*\delta \hookrightarrow V$ is the inclusion. The only difference with the true total Abelian Lefschetz zeta function $\zeta_A(\pi^*\delta)$ (conveniently evaluated in $H_1(V) \oplus H_1(T_\phi)$) is the sign of the contribution of each orbit γ of ψ_δ , that in the π -twisted version is multiplied by the sign of the power of δ determined by $\pi_* \circ i_*([\gamma])$. Using the relation

in (4.8) we can understand this difference by expressing $\zeta_\pi(\phi)$ in terms of the “local” Reidemeister torsions of (V, T_ϕ, π, F) near the orbits of ϕ . Denote by

$$\tau_V(\pi^*\delta, t) \in \mathbb{Z}[[H_1(V)]][[t]]$$

the series obtained from $\tau_A(\pi^*\delta, t)$ by replacing the variables keeping track of classes $c \in H_1(\pi^*\delta)$ with the corresponding variables z_{c_V} (using Notation 4.11) encoding classes in $H_1(V)$.

Proposition 4.17. *Given a surface bundle (V, T_ϕ, π, F) , we have*

$$\zeta_\pi(\phi) = \prod_{\substack{\delta \text{ simple} \\ \text{in } T_\phi}} \begin{cases} \tau_V(\pi^*\delta, t_{[\delta]} \cdot t^{\langle \delta, S \rangle}) & \text{if } \delta \text{ elliptic;} \\ \frac{1}{\tau_V(\pi^*\delta, t_{[\delta]} \cdot t^{\langle \delta, S \rangle})} & \text{if } \delta \text{ positive hyperbolic;} \\ \frac{1}{\tau_V(\pi^*\delta, -t_{[\delta]} \cdot t^{\langle \delta, S \rangle})} & \text{if } \delta \text{ negative hyperbolic.} \end{cases}$$

Proof. We prove here only the case of δ negative hyperbolic, leaving to the reader the analogue computations for the other two cases. For the case in hand, by (4.1) we have $\varepsilon(\delta^m) = (-1)^{m+1}$ and, as in the proof of Theorem 4.15, we get

$$\zeta_\delta(\pi, t_{[\delta]} \cdot t^{\langle \delta, S \rangle}) = \prod_{i=0}^2 \left(\exp \left(\sum_{m \geq 1} (-1)^{m+1} \text{tr}(\rho_i^\pi([\delta^m])) \frac{t^{\langle \delta, S \rangle} \cdot m}{m} \right) \right)^{(-1)^i}.$$

By definition, $\rho_i^\pi([\delta^m])$ is the matrix induced on $H_i(\tilde{F}, \mathbb{Z}[H_1(V) \oplus H_1(T_\phi)])$ (via the evaluation induced by $i_* \oplus (\pi \circ i)_*$ on the variables) by a representative $\psi_{[\delta^m]}$ of $\mathbf{m}([\delta^m])$. Then, using the Taylor series of the formal logarithm \log and the linear algebra relation $\exp(\text{tr}(\log(\cdot))) = \det(\cdot)$, we get:

$$\begin{aligned} \zeta_\delta(\pi, t_{[\delta]} \cdot t^{\langle \delta, S \rangle}) &= \prod_{i=0}^2 \left(\exp \left(\sum_{m \geq 1} -\text{tr}(\rho_i^\pi([\delta^m])) \frac{(-t^{\langle \delta, S \rangle})^m}{m} \right) \right)^{(-1)^i} \\ &= \prod_{i=0}^2 \left(\exp \left(\text{tr} \left(- \sum_{m \geq 1} \frac{(\rho_i^\pi([\delta]) \cdot (-t^{\langle \delta, S \rangle}))^m}{m} \right) \right) \right)^{(-1)^i} \\ &= \prod_{i=0}^2 \left(\exp \left(\text{tr} \left(\log \left(\mathbb{1} - \rho_i^\pi([\delta]) \cdot (-t^{\langle \delta, S \rangle}) \right) \right) \right) \right)^{(-1)^i} \\ &= \prod_{i=0}^2 \left(\det \left(\mathbb{1} - \rho_i^\pi([\delta]) \cdot (-t^{\langle \delta, S \rangle}) \right) \right)^{(-1)^i} \\ &= \left(\tau_V(\pi^*\delta, -t_{[\delta]} \cdot t^{\langle \delta, S \rangle}) \right)^{-1} \end{aligned}$$

where the last equivalence follows by applying the relation (4.8) to $\pi^*\delta$ (with monodromy $\mathbf{m}([\delta])$) and replacing as usual the variables encoding the classes $c \in H_1(\pi^*\delta)$ with the variables $z_{i_*(c)} \cdot t_{\pi_* \circ i_*(c)}$. \square

Observe that the weight with which we count a simple orbit δ is, up to eventually taking the reciprocal, just an evaluation of the standard total Abelian Reidemeister torsion of T_{ψ_δ} .

Example 4.18 (The trivial example). Consider the disk bundle $(\mathbb{D} \times T_\phi, T_\phi, \pi, \mathbb{D})$. For any simple orbit δ , $\pi^*\delta$ is then a solid torus, whose Reidemeister torsion is

$$\tau(\mathbb{D} \times S^1, t) = \frac{1}{1-t}.$$

Since here $\pi_* : H_1(V) \rightarrow H_1(T_\phi)$ is an isomorphism, we can set all the variables $z_b = 1$ in $\zeta_\pi(\phi)$ without losing information. Then Proposition 4.17 gives

$$\zeta_\pi(\phi) = \prod_{\substack{\delta \text{ simple} \\ \text{in } T_\phi}} \begin{cases} \tau(\pi^*\delta, t_{[\delta]} \cdot t^{\langle \delta, S \rangle}) = \frac{1}{1 - t_{[\delta]} \cdot t^{\langle \delta, S \rangle}} & \text{if } \delta \text{ ell.}; \\ \frac{1}{\tau(\pi^*\delta, t_{[\delta]} \cdot t^{\langle \delta, S \rangle})} = 1 - t_{[\delta]} \cdot t^{\langle \delta, S \rangle} & \text{if } \delta \text{ pos. hyp.}; \\ \frac{1}{\tau(\pi^*\delta, -t_{[\delta]} \cdot t^{\langle \delta, S \rangle})} = 1 + t_{[\delta]} \cdot t^{\langle \delta, S \rangle} & \text{if } \delta \text{ neg. hyp.} \end{cases}$$

which equals the total Abelian Reidemeister torsion of T_ϕ (cf. equations (4.2), (4.3) and (4.8)).

Definition 4.19. Given a 3-manifold Y , we say that a representation $\rho : \pi_1(Y) \rightarrow \text{GL}(n, R)$ over an Abelian ring R is a *(surface) bundled representation* if there exist a smooth surface bundle (V, Y, π, F) and a group homomorphism

$$h : \mathbb{Z}[H_1(V) \oplus H_1(T_\phi)] \longrightarrow R$$

such that ρ is conjugate with $h(\rho_i^\pi)$ for some $i \in \{0, 1, 2\}$, where ρ_i^π is the total Abelian monodromy representation of $\pi_1(Y)$ of index i induced by π .

Example 4.20. Given a surface bundle (V, Y, π, F) with F closed of genus g , the algebraic monodromy maps (defined in (4.12))

$$\mathbf{m}_i^\pi : \pi_1(Y) \longrightarrow \text{GL}(r_i, \mathbb{Z}).$$

with $r_i = \text{rank}(H_i(F, \mathbb{Z}))$ are clearly bundled representations, since for any $\alpha \in \pi_1(Y)$, $\mathbf{m}_i^\pi(\alpha) = h(\rho_i^\pi(\alpha))$ with

$$\begin{array}{ccc} h : \mathbb{Z}[H_1(V) \oplus H_1(Y)] & \longrightarrow & \mathbb{Z} \\ n z_b t_a & \longmapsto & n \end{array}$$

for any $b \in H_1(V)$ and $a \in H_1(Y)$. On the other hand it is a classical result (see for example [4]) that the map

$$(V, Y, \pi, F) \longmapsto \mathbf{m}_\pi \in \text{MCG}(F)$$

that associates to a closed surface bundle the corresponding monodromy homomorphism (4.11) induces a bijection between the set of bundle isomorphism classes over Y with fiber F and the set of conjugacy classes of representations of $\pi_1(Y)$ into $\text{MCG}(F)$.

These facts, together with the well known surjection

$$\begin{array}{ccc} \text{MCG}(F) & \twoheadrightarrow & \text{GL}(H_1(F, \mathbb{Z})), \\ \psi & \longmapsto & \psi_*|_{H_1(F, \mathbb{Z})} \end{array}$$

imply that, for any Y and $g \geq 1$ all representations of $\pi_1(Y)$ over $\text{GL}(2g, \mathbb{Z})$ are bundled.

Observe that the total Abelian Reidemeister torsion of T_ϕ as defined in (4.8) is the product of three factors. The i -th *total Abelian Alexander polynomial* of T_ϕ is

$$\Delta_i(T_\phi, t) = \det \left(\mathbb{1} - t(t_\mu \tilde{\phi}_i) \right).$$

Given an orbit δ of ϕ , define $\Delta_{iV}(\pi^*\delta, t) \in \mathbb{Z}[[H_1(V)]][[t]]$ by evaluating in $H_1(V)$ the variables (encoding elements in $H_1(\pi^*\delta)$) of $\Delta_i(T_\phi, t)$ as usual, so that

$$\tau_V(\pi^*\delta, t) = \prod_{i=0}^2 (\Delta_{iV}(T_\phi, t))^{(-1)^{i+1}}.$$

Corollary 4.21. *Let R be an Abelian ring and $\rho : \pi_1(T_\phi) \rightarrow \mathrm{GL}(n, R)$ be a bundled representation which is conjugate with $h(\rho_i^\pi)$ for some $i \in \{0, 1, 2\}$, bundle (V, T_ϕ, π, F) and homomorphism h as in Definition 4.19. Then:*

$$\tau_\rho(T_\phi, t) = \prod_{\substack{\delta \text{ simple} \\ \text{in } T_\phi}} \begin{cases} h(\Delta_{iV}(\pi^*\delta, t_{[\delta]} \cdot t^{\langle \delta, S \rangle})) & \text{if } \delta \text{ ell.}; \\ \frac{1}{h(\Delta_{iV}(\pi^*\delta, t_{[\delta]} \cdot t^{\langle \delta, S \rangle}))} & \text{if } \delta \text{ pos. hyp.}; \\ \frac{1}{h(\Delta_{iV}(\pi^*\delta, -t_{[\delta]} \cdot t^{\langle \delta, S \rangle}))} & \text{if } \delta \text{ neg. hyp.} \end{cases}$$

Remark 4.22. Theorem 4.15, Proposition 4.17 and Corollary 4.21 give an answer to Question 2 of the introduction, at least for bundled representations of fibered 3-manifolds. Essentially, for a bundled representation ρ of $\pi_1(T_\phi)$ there exists a surface bundle over T_ϕ for which we can describe the corresponding Lin's non-Abelian Reidemeister torsion $\tau_\rho(T_\phi)$ in terms of the “local” Alexander polynomials of the mapping tori (induced by the bundle) above the simple orbits of ϕ . Each of these mapping tori can be considered as a *local model* for the corresponding orbit δ that encodes in its topology informations about the free homotopy class of δ .

Remark 4.23. Here we do not completely study the family of the bundle representations of $\pi_1(Y)$. Example 4.20 (and its analogue for bundles with fibers with boundary) gives a relatively big class of examples. We observe that this class can be easily enlarged by using the representations $\tilde{\mathbf{m}}_i^V$ (and not just the weaker \mathbf{m}_i) and composed with some representation of $H_1(V)$ that is non-trivial on $\ker(\pi_*|_{H_1(V)})$.

4.5. Twisted GT and Lefschetz zeta functions.

For this subsection, fix a closed symplectic surface (S, Ω) of genus g and a mapping class $\mathbf{m} \in \mathrm{MCG}(S)$. Let ϕ be a symplectic representative of \mathbf{m} and consider the 4-manifold

$$X_\phi := S^1 \times T_\phi.$$

X_ϕ naturally fibers over a torus with fiber S and can be endowed with a symplectic form ω given by Thurston's Theorem 3.1:

$$(4.13) \quad \omega := ds \wedge dt + \Omega_{X_\phi},$$

where:

- s and t are coordinates for S^1 and, respectively, $\frac{[0, +\infty)}{t \sim t+1}$ in T_ϕ (with a slight abuse of notation, we identified the coordinates $(s, t) \in S^1 \times S^1$ of the base torus with the corresponding coordinates in X_ϕ);

- Ω_{X_ϕ} is a closed 2-form on X_ϕ that restricts to Ω over each fiber F .

We hope that the use of the letter t to denote both the coordinate for $[0, +\infty)$ and the formal variable of Lefschetz zeta functions and Reidemeister torsions will not induce confusion.

Theorem 4.24 ([15],[13]). *If the genus of S is $g \geq 2$ then*

$$\text{GT}(X_\phi, \omega) = \zeta_A(\phi)|_{t=1}.$$

Sketch of the proof. Let J be an ω -compatible almost complex structure on X_ϕ such that $J\partial_s = \partial_t$. Using the condition $g \geq 2$, one can show that, for any $A \in H_2(X_\phi)$, $d_A \geq 0$ if and only if A is a Künneth product $A = [S^1] \times a$ for some $a \in H_1(T_\phi)$, in which case $d_A = 0$. If δ is any periodic orbit of ϕ , the surface $S^1 \times \delta$ is J -holomorphic and these are in fact all the J -holomorphic curves in (X_ϕ, J) that we need to consider (cf. [13, Lemma 2.6]). Clearly the curve

$$C_\delta := S^1 \times \delta$$

is embedded if and only if δ is simple. Moreover it is possible to prove that C_δ is regular (so that $\epsilon(C_\delta)$ is well defined) if and only if δ is non-degenerate, in which case

$$\epsilon(C_{\delta^n}) = \epsilon(\delta^n)$$

for any $n > 0$ (see [13, Lemma 2.7] for details). We have then the following exhaustive correspondence:

$$(4.14) \quad \begin{array}{ll} C_\delta \text{ of type } (+, 0) & \longleftrightarrow \delta \text{ elliptic,} \\ C_\delta \text{ of type } (-, 0) & \longleftrightarrow \delta \text{ positive hyperbolic,} \\ C_\delta \text{ of type } (+, 1) & \longleftrightarrow \delta \text{ negative hyperbolic.} \end{array}$$

Comparing (4.2) and (2.5) normalized by (2.6) we have then

$$P(C_\delta, t) = \zeta_\delta(t)$$

and, with the natural identification $t_{[C_\delta]} \equiv t_{[\delta]}$, we obtain:

$$\text{GT}(X_\phi, \omega) = \prod_{C_\delta \text{ embedded}} P(C_\delta, t_{[C_\delta]}) = \prod_{\delta \text{ simple}} \zeta_\delta(t_{[\delta]}) = \zeta_A(\phi)|_{t=1}.$$

□

The following is essentially Theorem 1.2 in introduction.

Theorem 4.25. *Let (V, T_ϕ, π, F) be a smooth surface bundle with F closed and let $(S^1 \times V, X_\phi, \Pi, F)$ be the natural bundle induced by the multiplication by S^1 , where $\Pi := \text{Id} \times \pi$. Endow X_ϕ with the symplectic form ω given in (4.13) and suppose that both F and S have genus greater than 1. Then:*

$$\text{GT}_\Pi(X_\phi, \omega) = \zeta_\pi(\phi)|_{t=1}.$$

Proof. By definition of $\text{GT}_\Pi(X_\phi, \omega)$ and the exhaustive correspondence (4.14) we have

$$\text{GT}_\Pi(X_\phi, \omega) = \prod_{\delta \text{ simple}} P_\Pi(C_\delta).$$

It is then enough to prove that, for any δ :

$$(4.15) \quad P_\Pi(C_\delta) = \zeta_\delta(\pi, t_{[\delta]})$$

with the two identifications

$$(4.16) \quad \begin{aligned} t_{[S^1] \times a} &\equiv t_a && \text{for any } a \in H_1(T_\phi) \\ z_{[S^1] \times b} &\equiv z_b && \text{for any } b \in H_1(\pi^* \delta). \end{aligned}$$

between the variables in the definitions 3.7 - 3.8 and those in definitions 4.12 - 4.13.

For any δ simple fix first a symplectic form Ω_δ on F and a symplectic representative $\psi_\delta : F \rightarrow F$ of $\mathfrak{m}_\pi([\delta]) \in \text{MCG}(F)$. Then we can endow

$$\Pi^* C_\delta = S^1 \times \pi^* \delta \cong X_{\psi_\delta}$$

with a symplectic form ω_δ like in (4.13) that, being of the form (3.3), can be used to compute $P_\Pi(C_\delta)$. We can then check Equation (4.15) case by case using the correspondence in (4.14). If δ is elliptic (4.15) is satisfied since

$$P_\Pi(C_\delta) = \text{GT}_{X_\phi}(\Pi^* C_\delta) = \tau_V(\pi^* \delta, t_{[\delta]}) = \zeta_\delta(\pi, t_{[\delta]}),$$

where the second equality follows by applying Theorem 4.24 to $\Pi^* C_\delta \cong X_{\psi_\delta}$ with the identifications (4.16) and the equivalence (4.8), while the last equality comes from Proposition 4.17. The proof for δ positive hyperbolic works in a completely analogous way by just taking the reciprocals in the last equation.

Finally, suppose that δ is negative hyperbolic. Then the double cover ι of C_δ for which $\epsilon((C_\delta)_\iota) = -1$ “doubles C_δ in the t -direction of X_ϕ ”, so that

$$P_\Pi(C_\delta) = \frac{\text{GT}_{X_\phi}(\Pi^* C_\delta)}{\text{GT}_{X_\phi}(\Pi^* C_{\delta^2})}.$$

Observing that $\Pi^* C_{\delta^2} \cong X_{\psi_\delta^2}$, reasoning as in the elliptic case we get

$$P_\Pi(C_\delta) = \frac{\tau_V(\pi^* \delta, t_{[\delta]})}{\tau_V(\pi^* \delta^2, t_{[\delta]}^2)}$$

where $\tau_V(\pi^* \delta^2, t)$ is obtained from $\tau_A(T_{\psi_\delta^2}, t) = \tau_A(\pi^* \delta^2, t)$ by replacing the variables $z_b, b \in H_1(\pi^* \delta^2)$, with variables z_{b_V} where $b_V \in H_1(V)$ is the image of b via the composition of the homomorphisms $H_1(\pi^* \delta^2) \rightarrow H_1(\pi^* \delta)$ (induced by the double cover projection) and $H_1(\pi^* \delta) \rightarrow H_1(V)$ (induced by the inclusion). Then:

$$\begin{aligned} P_\Pi(C_\delta) &= \frac{\prod_{i=1}^2 (\det(\mathbb{1} - t_{[\delta]}(\tilde{\mathfrak{m}}_i^V)))^{(-1)^{i+1}}}{\prod_{i=1}^2 (\det(\mathbb{1} - t_{[\delta]}^2(\tilde{\mathfrak{m}}_i^V)^2))^{(-1)^{i+1}}} \\ &= \prod_{i=1}^2 \left(\frac{\det(\mathbb{1} - t_{[\delta]}(\tilde{\mathfrak{m}}_i^V))}{\det(\mathbb{1} - t_{[\delta]}(\tilde{\mathfrak{m}}_i^V)) \cdot \det(\mathbb{1} + t_{[\delta]}(\tilde{\mathfrak{m}}_i^V))} \right)^{(-1)^{i+1}} \\ &= \prod_{i=1}^2 \left(\frac{1}{\det(\mathbb{1} - (-t_{[\delta]})(\tilde{\mathfrak{m}}_i^V))} \right)^{(-1)^{i+1}} \\ &= \frac{1}{\tau_V(\pi^* \delta, -t_{[\delta]})} = \zeta_\delta(\pi, t_{[\delta]}). \end{aligned}$$

□

The last theorem, together with Theorem 4.15 and Corollary 4.21, gives an answer to Question 1 in the introduction for twisted Reidemeister torsions associated with surface bundles or bundled representations. We end the paper with the following two natural questions. Given a 4-dimensional symplectic manifold (X, ω) , fix a smooth surface bundle (W, X, π, F) with F closed and $[F] \neq 0$ in $H_2(W, \mathbb{R})$:

- (1) Is $\text{GT}_\pi(X, \omega)$ independent (eventually up to global shifts on the variables) on the choice of ω ?
- (2) Observing that W can always be endowed with a symplectic form ω_W (given by Thurston's theorem), is there any relation between $\text{GT}_\pi(X, \omega)$ and the higher-dimensional Gromov series of (W, ω_W) ?

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